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Realization Theory of Discrete-Time Dynamical Systems

With 14 Figures

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Preface

In this monograph, we will present a basis for the Realization Theory of Discrete-time Dynamical Systems. From the view of input/output data in the discrete-time case, mathematical models will be constructed. We will propose for the first time some dynamical systems, which are General Dynamical Systems, Linear Representation Systems, Affine Dynamical Systems, Pseudo Linear Systems, Almost Linear Systems and So-called Linear Systems for discrete-time. We will also solve realization problems for the systems. This monograph includes new results and constructs a new and very wide inclusion relation for various non-linear dynamical systems. The relation is displayed in the enclosed figure where our systems are written in italic fonts.

In the case of continuous-time, General Dynamical Systems, Linear Representation Systems and Affine Dynamical Systems are discussed in Realization Theory of Continuous-time Dynamical Systems by T. Matsuo, Lecture Notes in Control and Information Science, Vol. 32, Springer, 1981.

Regarding the dynamical systems discussed in the book, after establishing some special features in discrete-time, this monograph will present new results, which have not previously been obtained. It will also show more concrete and practical results. Hence, this monograph is an expansion on discrete-time and a new development. Moreover, it will newly propose Pseudo Linear Systems, Almost Linear Systems and So-called Linear Systems. Therefore, this monograph will be a trial for the organization of various dynamical systems.

Realization problems can be roughly stated as follows:

A. Finding a mathematical model (equivalently, dynamical system) from an input/output relation of a given black-box.

B. If possible, clarifying when the mathematical model can be actually embodied. Specifically, investigating when the mathematical model can be finite dimensional.

C. Seeking out the mathematical model from finite input/output data of the black-box. This problem may be called a partial realization problem.

The definition of the General Dynamical Systems was proposed in 1960, while the dynamical system theory was neatly established in 1969 by R. E. Kalman, only for discrete-time linear systems in the sense of algebraic system theory. Based on this, T. Matsuo established the realization theory of general dynamical systems for continuous-time.

On the other hand, automata's theory has been independently developed since about 1959. It was recognized that there was a close relationship between it and the realization theory; however, no clear relationship between them could be found. We will show that an automaton is a special feature of our discrete-time general dynamical systems.

We will introduce General Dynamical Systems, Linear Representation Systems, Affine Dynamical Systems, Pseudo Linear Systems, Almost Linear Systems and So-called Linear Systems for discrete-time and demonstrate the relationship between them and the other dynamical systems.

The monograph is intended for graduate students and researchers who study or research on the control theory (or simulation).

Professor Matsuo, who is one of the authors, died in April, 1993. With suggestions from Professor R. E. Kalman, he had been mainly studying the realization theory against the circumstance where science seems hardly to grow. This monograph is a part of his works.

I thank Professor R. E. Kalman who gave me stimuli to research these realization problems.

I very much thank Ms. Yoko Sugiyama and Ms. Helen Kyle who understood our scientific contents as much as possible, and made the first manuscript, which had many errors in English, into a very readable manuscript.

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1 Introduction

The realization problem that we will state here can be divided into the following three problems A, B and C. Where I/O is the set of input/output maps that may be the input/output relation of a given black-box. CD is the category of dynamical systems which may have the same behavior (equivalently, input/output relation) as the black-box.

A. The existence and uniqueness in algebraic sense.

For any input/output map $a \in I/O$, find out at least one dynamical system $\sigma \in CD$ such that the behavior of it is a. Also prove that any two dynamical systems that have the same behavior a are isomorphic in the sense of the category CD.

B. The finite dimensionality of the dynamical systems.

Clarify when a dynamical system $\sigma \in CD$ is finite dimensional. Because finite dimensional dynamical systems are actually appearing by linear (or non-linear) circuits or computer programs, it is very important that these conditions become clear.

C. Deriving the dynamical systems from finite data.

Partial realization problems are to find the minimal dynamical system fit to a given finite input/output's data and to clarify when the minimal dynamical systems are isomorphic.

This realization problem was presented by R. E. Kalman in about 1960, and the realization problem for the linear system was neatly established by him in algebraic sense. Based on this idea, T. Matsuo established a basis of realization problem of non-linear system for continuous-time case. On the basis of these ideas, we present a basis of non-linear dynamical systems for discrete-time case.

The discrete-time dynamical systems become more important because of computer developments and mathematical programming. In this sense, discrete-time linear systems have many fruitful achievements. Discrete-time non-linear dynamical systems will try to have the same achievements. R. E. Kalman developed the linear system theory by using algebraic theory. Therefore, the algebraic theory will present much materials for non-linear dynamical system's developments.

Our mathematical models for a given black-box are said to be dynamical systems. In the other field, the word "*dynamical system*" is used. See Birkhoff [1927] and Hirsch and Smale [1974]. The "*dynamical system*" comes from classic astronomy. "*Dynamical system*" is a study of the topological prop-

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erties of free motions in non-linear systems without input and output mechanisms. Many researchers are working on the field.

On the other hand, R. E. Kalman first defined dynamical systems that have input and output mechanisms in 1963. Kalman claimed that his definition is "modern" and "*dynamical system*" is "classic". It may depend on that the dynamical systems present new notions and they include the "*dynamical systems*" without input and output mechanisms.

There are similarities between realization problem and physics. Checking if a physical model that represents a given physical phenomenon fits to Newton's, Einstein's, Kirchhoff's, Maxwell's or Schrodinger's law, a physicist wants to obtain the model without affecting input and output mechanisms. If the physical model fits to none of them, then the physicist insists that the model is wrong. He always checks that the model is consistent with one of these laws. The dynamical systems are only intended to satisfy the causality condition. From this point of view, the dynamical system may be more "modern" than the "*dynamical system*".

Our realization theory will present a basis of discrete-time dynamical systems; therefore, a style of this monograph may be unified. This monograph contains six chapters, except Chapters 1 and 2, which contain different dynamical systems. The composition of the chapters contains mainly three parts, except Chapter 3 and Chapter 5.

The first part contains the existence and uniqueness of the dynamical system to be considered.
The second part contains its finite dimensionality.
The third part contains its partial realization problem.
Finally, every chapter contains an Appendix that is prepared for proofs of the results listed before. Especially, for special dynamical systems discussed in Chapter 6 and 7, the fourth part may contain the real-time partial realization problem (equivalently, partial realization problem by single-experiment).

Our realization theory presents various dynamical systems, which are General Dynamical Systems, Linear Representation Systems, Affine Dynamical Systems, Pseudo Linear Systems, Almost Linear Systems and So-called Linear Systems (equivalently, linear systems with a non-zero initial state). We will also present a system of many dynamical systems in Figure 1.1. The presented dynamical systems are written in italic fonts in the figure.

Sontag [1979a b] presented a realization theorem for wide dynamical systems to be said to be state-affine systems, which contain linear, multi-linear, homogeneous and inhomogeneous bilinear systems, etc. However, he did not

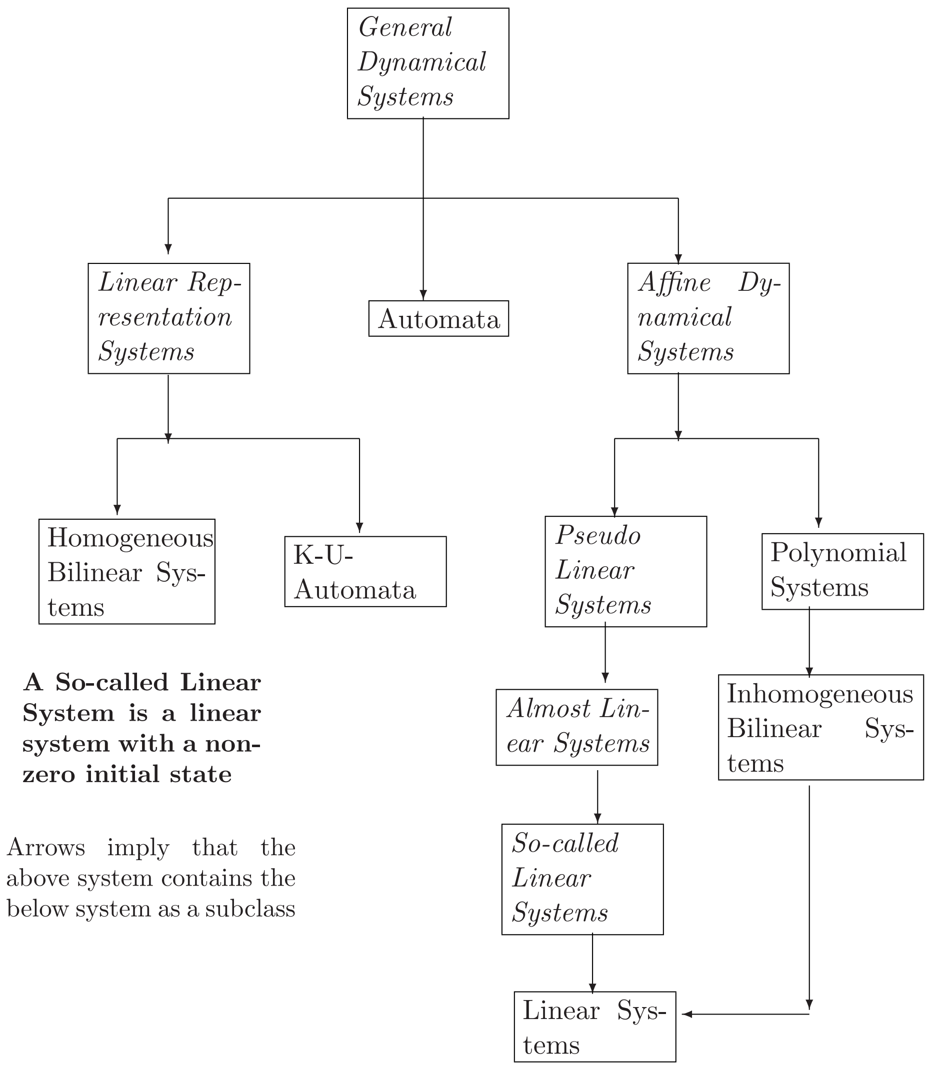

Fig. 1.1. An inclusion relation for various dynamical systems

give an initial object of the category. Then he concretely investigated the polynomial systems.

Contents of this monograph can be concretely stated as follows:

In Chapter 2 we present a set of experiments that can be allowed to a black box. The set that satisfies an axiom is said to be a concatenation monoid. Introducing an axiom of discrete-time which relax the axiom of continuous-time, we will show that the concatenation monoid is unique. Note that there exist some concatenation monoids in continuous-time case. Next, we present the Representation Theorem (2.6) for any input/output map with causality. The theorem is rewritten for discrete-time case by using the one of continuous-time. The theorem says that any input/output map can be characterized by an input response map.

In Chapter 3 let the set of output values Y be any set. Then we present General Dynamical Systems that are mathematical models for any input response map. Let I/O be the set of any input response maps and CD be the category of canonical (reachable and distinguishable) General Dynamical Systems. Then we obtain the existence and uniqueness theorem. Moreover, we discuss the finiteness of the General Dynamical Systems. We give a criterion for finiteness of the system. We give a procedure to obtain the system from a given input response map.

In Chapter 4 let the set of output values Y be any linear space over the field K. And let I/O be the set of any input response map and CD be the category of canonical (quasi-reachable and distinguishable) Linear Representation Systems. Then we obtain the existence and uniqueness theorem.

Moreover, we investigate details of finite dimensional Linear Representation Systems. We give a criterion for the canonicality of finite dimensional Linear Representation Systems. In the isomorphic classes of finite dimensional canonical Linear Representation Systems, there exists a unique quasi-reachable standard system and a unique distinguishable standard system. It is also shown that the following three conditions are equivalent:

1) An input response map is the behavior (input/output relation) of a finite dimensional Linear Representation System.

2) The rank of infinite Hankel matrix is finite.

3) An input response map is rational.

Also a procedure to obtain the quasi-reachable standard system from an input response map is given.

Moreover, the partial realization problem for the dynamical systems is discussed by multi-experiment. For a partial input response map, there exists a minimum Linear Representation System with the same behavior. Generally, the minimum partial realizations are not unique up to isomorphism. To solve the uniqueness problem for a partial realization problem, we introduce the notion of natural partial realization. Then the following results are obtained:
1) A criterion for the existence of the natural partial realizations is given by the rank condition of finite Hankel matrix.
2) The existence condition of the natural partial realizations is equivalent to the uniqueness condition of minimum partial realizations modulo isomorphism.
3) An algorithm to obtain a natural partial realization from a given partial input response map is given.

In Chapter 5 let the set of output values Y be any linear space over the field K. And let I/O be the set of any input response map and CD be the category of canonical (quasi-reachable and distinguishable) Affine Dynamical Systems. Then we obtain the existence and uniqueness theorem.
We investigate details of finite dimensional Affine Dynamical Systems and we obtain the same results as in Linear Representation Systems. We list the results as follows. A criterion for canonicality of finite dimensional Affine Dynamical Systems is given. In the isomorphic classes of finite dimensional canonical Affine Dynamical Systems, there exists a unique quasi-reachable standard system and a unique distinguishable standard system. It is also showed that the following two conditions are equivalent:
1) An input response map is the behavior (input/output relation) of a finite dimensional Linear Representation System.
2) The rank of infinite Hankel matrix is finite.

A procedure to obtain the quasi-reachable standard system from an input response map is given.

In Chapter 6 let the set of output values Y be any linear space over the field K. And let I/O be the set of any time-invariant input response map (equivalently, any input/output map with causality and time-invariance) and CD be the category of canonical (quasi-reachable and observable) Pseudo Linear Systems. We give a representation theorem of the time-invariant input response map, which says that any time-invariant input response map can be characterized by a modified impulse response. It is also shown that any input/output relations with causality and time-invariance can be expressed as a convolution of inputs and the modified impulse responses. This implies that the modified impulse response is an extension of the impulse response in linear systems.

Also we obtain the existence and uniqueness theorem. The Pseudo Linear Systems are very important dynamical systems because there are non-linear circuits with FET transistor as an example of them.

Moreover, we investigate details of finite dimensional Pseudo Linear Systems. A criterion for the canonicality of finite dimensional Pseudo Linear Systems is given. In the isomorphic classes of finite dimensional canonical Pseudo Linear Systems, there exists a unique quasi-reachable standard system and a unique observable standard system. It is also shown that the following two conditions are equivalent:
1) A time-invariant input response map is the behavior (input/output relation) of a finite dimensional Pseudo Linear System.
2) The rank of infinite Input/Output Matrix is finite.
A procedure to obtain the quasi-reachable standard system from a time-invariant input response map is given.

Moreover, the partial realization problem for the dynamical systems is discussed by multi-experiment. For a partial input response map, there exists a minimum Pseudo Linear System with the same behavior. Generally, the minimum partial realizations are not unique up to isomorphism. To solve the uniqueness problem for partial realization problem, we introduce the notion of natural partial realization. Then the following results are obtained:
1) A criterion for the existence of the natural partial realizations is given by the rank condition of finite Input/Output Matrix.
2) The existence condition of the natural partial realization is equivalent to the uniqueness condition of minimum partial realizations modulo isomorphism.
3) An algorithm to obtain a natural partial realization from a partial time-invariant input response map is given.

For the Pseudo Linear Systems, we can easily discuss a real time partial realization problem. The reason comes from the time-invariance of input/output relation. If we know that a physical object to be considered is finite dimensional Pseudo Linear System and less than L dimensional, we give an algorithm to obtain a quasi-reachable standard system from real time partial data (real time data).

In Chapter 7 let the set of output values Y be any linear space over the field K. And let I/O be the set of any time-invariant, affine input response map (equivalently, any input/output map with causality, time-invariance and affinity) and CD be the category of canonical (quasi-reachable and observable) Almost Linear Systems. We give a representation theorem of the time-invariant, affine input response map, which says that any time-invariant, affine input response map can be characterized by a modified impulse response. It is also shown that any input/output relations with causality and time-invariance and affinity can be expressed as a convolution of inputs and the modified impulse responses. This implies that the modified impulse response is an extension of the impulse response in linear systems. This is the same result as in Pseudo Linear Systems.

We obtain the existence and uniqueness theorem. The Almost Linear Systems are very important dynamical systems because there are So-called Linear Systems as an example of them. The So-called Linear Systems are linear systems with a non-zero initial state. We can also show an Almost Linear System which is not a linear system with a non-zero initial state.

Moreover, we investigate details of finite dimensional Almost Linear Systems. A criterion for the canonicality of finite dimensional Almost Linear Systems is given. In the isomorphic classes of finite dimensional canonical Almost Linear Systems, there exists a unique quasi-reachable standard system and a unique observable standard system. It is also shown that the following two conditions are equivalent:

1) A time-invariant, affine input response map is the behavior (input/output relation) of a finite dimensional Almost Linear System.

2) The rank of infinite Input/Output Matrix is finite.

A procedure to obtain the quasi-reachable standard system from a time-invariant, affine input response map is given.

Moreover, the partial realization problem for the dynamical systems is discussed by multi-experiment. For a partial time-invariant, affine input response map, there exists a minimum Almost Linear System with the same behavior. Generally, the minimum partial realizations are not unique up to isomorphism. To solve the uniqueness problem for the partial realization problem, we introduce the notion of natural partial realization. Then the following results are obtained:

1) A necessary and sufficient condition for the existence of the natural partial realizations is given by the rank condition of finite Input/Output Matrix.

2) The existence condition of the natural partial realization is equivalent to the uniqueness condition of minimum partial realizations modulo isomorphism.
3) An algorithm to obtain a natural partial realization from a partial time-invariant, affine input response map is given.

For the Almost Linear Systems, we can discuss real time partial realization problem. The reason comes from the time-invariance of the input/output relation. If we know that a physical object to be considered is finite dimensional Almost Linear System and less than L dimensional, we can give an algorithm to obtain a quasi-reachable standard system from real time partial data (real time data). By applying this to So-called Linear Systems, both the partial realization problem of the linear systems and state estimation problem will be solved simultaneously.

In Chapter 8 let the set of output values Y be any linear space over the field K. And let I/O be the set of special time-invariant, affine input response map (equivalently, any input/output map with causality, time-invariance and special affinity) and CD be the category of canonical (reachable and observable) So-called Linear Systems(equivalently, linear systems with a non-zero initial state). Discussing the relations between So-called Linear Systems and Almost Linear Systems, we make the relation clear. Then we clarify when a time-invariant, affine input response map is a behaviour of So-called Linear Systems.

Moreover, we investigate details of finite dimensional So-called Linear Systems and discuss a partial realization problem and real time partial realization problem for So-called Linear Systems. For these problems, we obtain the same results as in Almost Linear Systems.

[**Notations**]

N: the set of non-negative integers.
K: a field. $x \in X$: an element x belongs to a set X.
$F(X_1, X_2)$: the set of any function from a set X_1 to a set X_2.
$F(X)$: the set of any function from a set X to a set X.
$L(X_1, X_2)$: the set of any linear operator from a linear space X_1 to a linear space X_2. $L(X) := L(X, X)$.
$\mathbf{K}^n$: an n-dimensional coordinate space over the field **K**.
dom f: a domain of an operator f. im f: an image of an operator f.
ker f:=$\{x \in \text{dom } f \ ; \ f(x) = 0 \}$

2 Input/Output Maps

2.1 Concatenation Monoid: Set of Experiments

We define the set of all input sequences that can be applied to system after the present time (zero-time). We consider the set as a concatenation monoid. The concatenation monoid for continuous-time has been introduced in Matsuo [1981]. It is the set of experiments with the operation of concatenation which make two experiments in succession. In continuous-time case, there existed such sets, but we will show that there exists uniquely such a set in discrete-time case.

Let U be a set of input values and let $F_s(N^+$, U): $=\{\omega$; function $\omega : (0,n] \to U$ for some $n \in N\}$. We assume that *1* (the function defined on empty set (0,0]) belongs to $F_s(N^+,U)$. Then we can consider $F_s(N^+,U)$ as the set of words for the set U of an alphabet. $F_s(N^+,U)$ becomes the free monoid over U. Let $|\omega| := n$ denotes the length of an input sequence $\omega := \omega(n)\omega(n-1)\cdots\omega(2)\omega(1) \in F_s(N^+,U)$. We assume that $|1|$=0 for the empty word *1*. Now we define the following operation of concatenation:

(2.1) Definition
$F_s(N^+,U) \times F_s(N^+,U) \to F_s(N^+,U)$; $(\omega_2,\omega_1) \mapsto \omega_2|\omega_1$,

$$(\omega_2|\omega_1)(n) := \begin{cases} \omega_1 & 0 \le n \le |\omega_1| \\ \omega_2(n-|\omega_1|) & |\omega_1| < n \le |\omega_1| + |\omega_2| \end{cases}$$

Then $F_s(N^+,U)$ becomes a monoid with a unit element *1* under the concatenation's operation.

(2.2) Definition
If a subset Ω of $F_s(N^+,U)$ satisfies the following conditions, then Ω is said to be a concatenation monoid.
(1) Ω is closed under the following operators $C(n)$ and $S_l(n)$ for $n \in N$. Where $C(n)$ and $S_l(n)$ are said to be cutting's operator and left-shift's operator respectively.
a). $C(n) : \Omega \to \Omega$;$\omega \mapsto C(n)\omega$.
Where $(C(n)\omega)(m) = \omega(m)$ for $0 < m \le$ min $(n, |\omega|)$, and $|C(n)\omega| = min($ n$, |\omega|)$. $C(0)\omega = 1$.
b). $S_l(n) : \Omega \to \Omega$; $\omega \mapsto S_l(n)\ \omega$.
Where $(S_l(n)\ \omega)(m)$= $\omega(n+m)$ for $0 < m \le |\omega| - n$. $|S_l(n)\omega| = \max(|\omega| - n, 0)$.

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(2) There exist $n \in N$ and $\omega \in \Omega$ such that $\omega(n) = u$ for any $u \in U$.

Then we obtain the following proposition:

(2.3) Proposition
There exists uniquely the concatenation monoid Ω, and it is $F_s(N^+, U)$; it is also the free-monoid U^* over U.

[proof] There exists Ω such that Ω is a concatenation monoid because $F_s(N^+, U)$ is a concatenation monoid evidently.
Next we show the uniqueness of it. Let $\omega \in F_s(N^+, U)$, then the definition of the concatenation monoid implies that there exist n $\in N$ and $\omega' \in \Omega$ such that ω'(n)= ω(m) for any m $\in$ dom ω. Since Ω is closed under cutting's operator and left-shift's operator, ω(m) $= C(1)(S_l(n-1)\omega') \in \Omega$ holds. Moreover, $\omega = \omega(|\omega|)\cdots\omega(2)\omega(1) \in \Omega$ holds and Ω is a sub-monoid of $F_s(N^+, U)$, it follows that $\omega \in \Omega$. Therefore, $\Omega = F_s(N^+, U)$ holds.

(2.4) Example
Let U be a set of finite elements. For convenience, Let $U = \{\alpha, \beta\}$. Then Ω is the set of words generated by letters α and β. In Ω, there is *1* as word of length 0, and there are α, β as words of length 1. There are $\alpha|\alpha, \beta|\alpha, \alpha|\beta and \beta|\beta$ as words of length 2, and so on. This Ω can be expressed by the following Figure 2.1.

2.2 Input Response Maps (Input/Output Maps with Causality)

We will consider an expression's method for input/output relations of an object to be observed or controlled. It is a black-box to which any element of the concatenation monoid U* can be applied and whose output values are in a set of output values. For the continuous-time case, the representation theorem for any input/output map with causality had been given by Matsuo [1977] and [1981]. This theorem can be easily rewritten into the case of discrete-time.

Any element of the concatenation monoid can be fed into a system that is an object. Where the output value γ(n) of it at time n belongs to the output set Y. Then input/output relation of it can be expressed by a map $A^{\#}$:Ω $\rightarrow Fs(N, Y)$. Where $Fs(N, Y)$:=$\{$ a function γ : [0, n] $\rightarrow Y$; for some n $\in$ N$\}$ and dom $A^{\#}(\omega)$ = [0,$|\omega|$] for $\omega \in \Omega$.

1

α β

$\alpha|\alpha$ $\beta|\alpha$ $\alpha|\beta$ $\beta|\beta$

$\alpha|\alpha|\alpha$ $\beta|\alpha|\alpha$ $\alpha|\beta|\alpha$ $\beta|\beta|\alpha$ $\alpha|\alpha|\beta$ $\beta|\alpha|\beta$ $\alpha|\beta|\beta$ $\beta|\beta|\beta$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Fig. 2.1. The table of concatenation monoid Ω over the input set $U = \{\alpha, \beta\}$

(2.5) Definition
If $A^{\#}(\omega_2|\omega_1)(n) = A^{\#}(\omega_3|\omega_1)(n)$, $0 \leq n \leq |\omega_1|$ for any ω_1, ω_2 and $\omega_3 \in \Omega$. Then the map $A^{\#}$ is said to be the input/output map with causality.
Let $Fc(\Omega, Fs(N,Y))$ be a set of the input/output maps with causality and $F(\Omega, Y)$ be a set of any function : $\Omega \to Y$.

(2.6) Theorem
A map $\alpha : Fc(\Omega, Fs(N,Y)) \to F(\Omega,Y)$ is bijective. Where $\alpha A^{\#}(\omega) = A^{\#}(\omega)(|\omega|)$ for $A^{\#} \in Fc(\Omega, Fs(N,Y))$ and $\omega \in \Omega$.

[proof] This theorem can be proved the same as Theorem (3.2) in Matsuo [1981]. Also see Matsuo [1987].

(2.7) Definition
An element a of $F(\Omega, Y)$ introduced in Theorem (2.6) is said to be an input response map.

Remark 1: Theorem (2.6) implies that an input/output map with causality is characterized by an input response map with the following equation $\gamma(|\omega|) = a(\omega)$. Where $\gamma(|\omega|)$ denotes an output value at the time $|\omega|$ for an input ω to have been ended to apply.
Remark 2: Based on the result that is similar to Theorem (2.6), Sontag [1979a] and [1979b] solved the discrete-time nonlinear realization problem of dynamical systems that are different from our dynamical systems.

Note that Sontag and Rouchaleau [1976] discussed nonlinear systems without referring kinds of input/output relations.

(2.8) Example
We will consider the case $U = \{\alpha, \beta\}$ considered in Example (2.4) and let a be an input response map, i.e. $a \in F(\Omega, Y)$. There is an $a(1)$ as output for an input of length 0, and there are $a(\alpha)$, $a(\beta)$ as outputs for inputs of length 1. There are $a(\alpha|\alpha)$, $a(\beta|\alpha)$, $a(\alpha|\beta)$ and $a(\beta|\beta)$ as outputs for inputs of length 2, and so on. This input response map a can be expressed as the following Figure 2.2.

$a(\mathit{1})$

$a(\alpha)$ $a(\beta)$

$a(\alpha|\alpha)$ $a(\beta|\alpha)$ $a(\alpha|\beta)$ $a(\beta|\beta)$

$a(\alpha|\alpha|\alpha)$ $a(\beta|\alpha|\alpha)$ $a(\alpha|\beta|\alpha)$ $a(\beta|\beta|\alpha)$ $a(\alpha|\alpha|\beta)$ $a(\beta|\alpha|\beta)$ $a(\alpha|\beta|\beta)$ $a(\beta|\beta|\beta)$

. . . .

Fig. 2.2. The table of an input response map $a \in F(\Omega, Y)$, where $U = \{\alpha, \beta\}$

2.3 Historical Notes and Concluding Remarks

The concatenation monoid in discrete-time is a rewritten one of continuous-time. It is a special feature of the discrete-time that there exists a unique concatenation monoid, namely, free monoid U^* of the set of input values. Note that there are several concatenation monoids in continuous-time. The uniqueness may produce fruitful results [see the relation between U-morphism and Ω-morphism in Chapter 3]. The concatenation monoid is the set of experimental means that can be applied to the black-box. Therefore, there is a close relation between it and classes of admissible controls introduced by Pontryagin [1963].

The Representation Theorem (2.6) for any input/output map with causality is the only discrete-time version of the theorem in continuous-time. In the same way as continuous-time, we probably can develop the realization theories of discrete-time dynamical systems owing to it.

3 General Dynamical Systems

Here we introduce General Dynamical Systems as suitable models for any input/output relation with causality (equivalently, any input response map). And we show that the systems are general systems.

Let the set of output values Y be any set through this chapter. The realization problem for General Dynamical Systems can be stated as follows:

[For a given input response map, there exist at least two General Dynamical Systems which realize (faithfully describe) it. Let σ_1 and σ_2 be General Dynamical Systems that have the same behavior, then σ_1 is isomorphic to σ_2 in the sense of General Dynamical System.]

Owing to by-products of this realization theory, we have the following. An automaton is a special General Dynamical System. The condition of minimal control system that includes the minimal automaton is equivalent to the condition of possibility for multiple experiment.

3.1 Realization Theory of General Dynamical Systems

Dynamical systems that have structures of state, input and output have been advocated by R. E. Kalman. Based on this idea, T. Matsuo had established the General Dynamical System of continuous-time (Matsuo [1981]).

In this section, we introduce the General Dynamical Systems of discrete-time. Firstly, we introduce the notion of canonical General Dynamical Systems. Then the solution of the realization problem for any non-linear system (equivalently, any input response map) will be given. Next, we discuss General Dynamical Systems with finite state structure. A criterion for the behavior of the canonical General Dynamical System is given. We give the procedure to obtain the canonical General Dynamical System from a given input response map. Also, we give a condition that canonical General Dynamical System is equivalent to an automaton.

In Appendix 3.5, we show how the General Dynamical Systems of discrete-time system are rewritten from ones of continuous-time system established by Matsuo.

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(3.1) Definition

A system given by the following equations is written as a collection $\sigma = ((X, F), x^0, h)$, and it is said to be a General Dynamical System.

$$\begin{cases} x(t+1) & = & F(\omega(t+1))x(t) \\ x(0) & = & x^0 \\ \gamma(t) & = & hx(t) \end{cases}$$

for any $t \in N$, where $x(t) \in X$ and $\gamma(t) \in Y$. Where X is a set that may be called a state set, F is an operator: U $\to F(X)$; $u \mapsto F(u)$, $x^0 \in X$ and $h : X \to Y$ is an operator.

The equation $x(t+1) = F(\omega(t+1)x(t)$ in the General Dynamical System may be said to be a U-action and may be written by (X, F), and x^0 is called an initial state.

The input response map $a_\sigma : \Omega \to Y$;$\omega \mapsto h\phi_F(\omega)x^0$ is said to be the behavior of σ. For an input response map $a \in F(\Omega, Y)$, σ that satisfies a_σ $= a$ is called a realization of a.

Where $\phi_F(\omega) = F(\omega(|\omega|))F(\omega(|\omega| - 1)) \cdots F(\omega(1))$.

A General Dynamical System σ is called reachable if the reachable set $\{\phi_F(\omega)x^0; \omega \in \Omega\}$ is equal to X and a General Dynamical System σ is called distinguishable if $h\phi_F(\omega)x_1^0 = h\phi_F(\omega)x_2^0$ for any $\omega \in \Omega$ implies $x_1 = x_2$.

A General Dynamical System σ is called canonical if σ is reachable and distinguishable.

Remark 1: The x(t) in the system equation of σ is the state that produces output values of a_σ at the time t, namely the state x(t) and an operator $h : X \to Y$ generates the output value $a_\sigma(\omega)$ at the time t by $a_\sigma(\omega) = hx(t)$, where t=$|\omega|$.

Remark 2: It is meant for σ to be a faithful model for the input response map a such that σ realizes a.

Remark 3: Notice that a canonical General Dynamical System $\sigma = ((X, F), x^0, h)$ is a system which has the most reduced state set X among systems that have the behavior a (see Definition (3-B.3), Proposition (3-B.5), Definition (3-C.1), Proposition (3-C.4), Definition (3-D.1), Proposition (3-D.7) in Appendix 3).

(3.2) Example

$(\Omega, *|)$ is a U-action by $u| : \Omega \to \Omega; \omega \mapsto u|\omega$. Let a unit element *1* in Ω be an initial state and $a \in F(\Omega, Y)$ be any response map. Then a collection $\sigma = ((\Omega, *|), 1, a)$ is a reachable General Dynamical System that realizes a.

(3.3) Example
Let $a \in F(\Omega, Y)$ be any input response map and S_l be defined by $S_l(u)a$ $:\Omega \to Y;\ \omega \mapsto a(\omega|u)$. Then $S_l \in F(F(\Omega, Y))$ for any $u \in U$. Hence the pair $(F(\Omega, Y), S_l)$ is a U-action. Let the input response map $a \in F(\Omega, Y)$ be an initial state and $\mathit{1}$ be a map: $F(\Omega, Y) \to Y;\ a \mapsto a(\mathit{1})$, then a collection $\sigma = ((F(\Omega, Y), S_{itl}), a, \mathit{1})$ is a distinguishable General Dynamical System which realizes a.

Remark: Examples (3.2) and (3.3) imply that there exist many General Dynamical Systems that realize a given input response. However, there is no relation between them. Therefore, we introduce the canonical General Dynamical Systems, and we will make the relation between them clear.

(3.4) Theorem
For any input response $a \in F(\Omega, Y)$, there exist the following two canonical General Dynamical Systems that realize it:
1) $\sigma = ((\Omega/_a, \tilde{\rceil}), [\mathit{1}], a^i)$.
Where $\Omega/_a$ is a quotient set derived by equivalence class $\omega = \omega' \leftrightarrow a(\hat{\omega}|\omega) = a(\hat{\omega}|\omega')$ for any $\hat{\omega} \in \Omega$. $\tilde{\rceil} : U \times \Omega/_a \to \Omega/_a;\ (u, [\omega]) \mapsto [u|\omega]$. $(\Omega/_a, \tilde{\rceil})$ is a U-action. $[\omega]$ denotes a set $\{\omega' \in \Omega; a(\hat{\omega}|\omega) = a(\hat{\omega}|\omega')$ for any $\hat{\omega} \in \Omega\}$. a^i is an injection accompanied with a.
2) $\sigma = ((S_l(\Omega)a, S_l), a, \mathit{1})$. Where $S_l(\Omega)a = \{S_l(\omega)a; \omega \in \Omega\}$.

[proof] This can be easily obtained by Corollary (3-B.4), Proposition (3-D.2) and Theorem (3-D.8).

We conclude that there exist the canonical General Dynamical Systems that realize any input response map in Theorem (3.4).
Next, we will insist the uniqueness of the systems that realize the same behavior.

(3.5) Definition
Let $\sigma_1 = ((X_1, F_1), x_1^0, h_1)$ and $\sigma_2 = ((X_2, F_2), x_2^0, h_2)$ be General Dynamical Systems. If a map $T : X_1 \to X_2$ satisfies $TF_1(u) = F_2(u)T$ for any u $\in U$, $Tx_1^0 = x_2^0$ and $h_1 = h_2T$, then T is said to be a dynamical system morphism $T : \sigma_1 \to \sigma_2$.
If the T is bijective then T is said to be an isomorphism.

(3.6) Realization Theorem
For any input response $a \in F(\Omega, Y)$, there exist at least two canonical General Dynamical Systems that realize it.
Let $\sigma_1 = ((X_1, F_1), x_1^0, h_1)$ and $\sigma_2 = ((X_2, F_2), x_2^0, h_2)$ be Dynamical Systems that realize an input response $a \in F(\Omega, Y)$, then there exists uniquely an isomorphism $T : \sigma_1 \to \sigma_2$.

[proof] The first part is equal to Theorem (3.4). The latter part is obtained by Remark of Theorem (3-D.14).

3.2 Finite General Dynamical Systems

Next we will discuss about the General Dynamical System $\sigma = ((X, F), x^0, h)$ with a finite state set X.

First we discuss the relation between the General Dynamical Systems and the complete automata.

(3.7) Example
Let the input value's set U be finite and a state set X be finite. Let the output set Y be $\{0, 1\}$ and h be a function $h : X \to Y$.
Here we will consider a General Dynamical System $\sigma = ((X, F), x^0, h)$. We will show that this σ is a complete automaton defined by Eilenberg.
Eilenberg has defined an automaton $A = (X, x^0, T)$ which satisfies the following five conditions:
1) U is a finite alphabet.
2) X is a finite state set.
3) $x^0 \in X$ is an initial state.
4) A composition : $U \times X \to X$; $(u, x) \mapsto x.u$ is a function.
5) $T \subseteq X$ is a target set. And the behavior of the automaton A is defined to be $|A| := \{s \in U^*; s \text{ is a successful path}\}$.
The condition 4) is equivalent to a U-action in our General Dynamical System. The conditions 1) to 3) are equivalent to the U-action with an initial state $((X, F), x^0)$.
Let the set T correspond to a characteristic function $\chi_T : X \to \{0, 1\}$ (if $\chi_T(x) = 1$ for $x \in T$ and $\chi_T(x) = 0$ for $x \notin T$), then this correspondence: $\hat{X} \to \{0, 1\}^X$ is bijective.
Where $\hat{X}$ is the power set of X. $\hat{X}$ can be considered as the set of target. Moreover, $\{0, 1\}^X$ can be considered as the set of any output map

$h : X \to Y$. If the automaton $A = (X, x^0, T)$ be corresponded to a General Dynamical System $\sigma = ((X, F), x^0, \chi_T)$, this correspondence is bijective. The ionverse correspondence of $\sigma = ((X, F), x^0, h)$ is an automaton $A = (X, x^0, h^{-1}(1))$. For the behavior $|A|$ of the automaton A and the behavior $a_\sigma : U^* \to \{0, 1\}$, there is a relation $a^{-1}(1) = |A|$. Therefore, we conclude that the complete automaton is a special system of the General Dynamical System.

(3.8) Definition
For an input response $a \in F(\Omega, Y)$, if there exists a General Dynamical System σ with a finite set X which realizes it, then a is said to be finite form.

(3.9) Proposition
For an input response $a \in F(\Omega, Y)$, the following conditions are equivalent:
1) a is a finite form.
2) The set $\Omega/_a$ of the quotient realization $((\Omega/_a, \tilde{|}), [\mathit{1}], a^i)$ is finite.
3) The set $S_l(\Omega)a$ of the image realization $((S_l(\Omega)a, S_l), a, \mathit{1})$ is finite.

[proof] This is trivial by Theorem (3.4).

(3.10) Example
We will consider the canonical General Dynamical System over $U = \{\alpha, \beta\}$. For an input response $a \in F(\Omega, Y)$, the subset canonical General Dynamical System is $((S_l(\Omega)a, S_l), a, \mathit{1})$. An input response map, when an input with length 0 has been fed is a. Input response maps when an input with length 1 have been fed are $S_l(\alpha)a$ and $S_l(\beta)a$. Input response maps when an input with length 2 have been fed are $S_l(\alpha|\alpha)a$, $S_l(\beta|\alpha)a$, $S_l(\alpha|\beta)a$ and $S_l(\beta|\beta)a$. Input response maps when an input with length 3 have been fed are $S_l(\alpha|\alpha|\alpha)a$, $S_l(\beta|\alpha|\alpha)a$, $\cdots$, and $S_l(\beta|\beta|\beta)a$, and so on. Hence the U-action $(S_l(\Omega)a, S_l)$ can represented in Figure 3.1.

(3.11) Theorem
Let the input value's set U be finite. To be simple, let $U = \{\alpha, \beta\}$. Let an input response $a \in F(\Omega, Y)$ satisfy Proposition (3.9), and let n := the number of independent elements in $S_l(\Omega)a$. Then the procedure to obtain $\sigma = ((X, F), x^0, h)$ with the behavior a is given by the following:

$$a$$

$$S_l(\alpha)a \qquad S_l(\beta)a$$

$$S_l(\alpha|\alpha)a \qquad S_l(\beta|\alpha)a \qquad S_l(\alpha|\beta)a \qquad S_l(\beta|\beta)a$$

$$S_l(\alpha|\alpha|\alpha)a \; S_l(\beta|\alpha|\alpha)a \; S_l(\alpha|\beta|\alpha)a \; S_l(\beta|\beta|\alpha)a \; S_l(\alpha|\alpha|\beta)a \; S_l(\beta|\alpha|\beta)a \; S_l(\alpha|\beta|\beta)a \; S_l(\beta|\beta|\beta)a$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Fig. 3.1. The U-action $(S_l)(\Omega)a, S_l)$ in Example (3.10)

1) Arrange a set $S_l(\Omega)a$ in the following order.
Arrange a, $S_l(\alpha)a$, $S_l(\beta)a$, and from left to right in the 3d row, i.e. $S_l(\alpha|\alpha)a$, $S_l(\beta|\alpha)a$, $S_l(\alpha|\beta)a$ and $S_l(\beta|\beta)a$.
Next, arrange from left to right in the fourth row, and so on.
Pull out n different elements in this order. Let them be a_1, a_2, $\cdots$, a_{n-1} and a_n. Where $a_1 := a$. (There always exist n different elements until the n-th row.)
2) Let a state set X be $\{1, 2, \cdots, n\}$ and an initial state x^0 be 1.
3) Let an output $h : X \to Y$ be $h(i) = a_i(1)$ for $i(1 \leq i \leq n)$.
4) Let a map $F : U \to X$ be $(F(\beta))(i) = j$. Where $S_l(\beta)a_i = a_j$ and $i(1 \leq i \leq n)$.
5) Let a map $F : U \to X$ be $(F(s))(i) = k$. Where $S_l(\beta)a_i = a_k$ and $i(1 \leq i \leq n)$.

[proof] See (3-D.15) in Appendix.

(3.12) Example
Let $U := \{-1, 0\}$ and $Y := R$. And let a considered black-box a be given by the following equation:
$\gamma(t+1) = \gamma^2(t) + \omega(t+1)$, $t \in N$, $\gamma(0) = 0$.
Where $\omega(t) \in U$ and $\gamma(t) \in Y$.

If we express an input response map a of this input/output relation in the same manner as Figure 2.2 in Example (2.8), we obtain the following Figure 3.2.
By applying Theorem (3.11), we obtain a canonical finite dynamical system $\sigma = ((X, F), x^0, h)$ which realizes a in the following manner:
1) Different elements in a set $S_l(\Omega)a$ are obviously $a_1 := a$, $a_2 := S_l(-1)a$ and $a_3 := S_l(0|-1)a$.
2) Let a state set X be $\{1, 2, 3\}$ and an initial state x^0 be 1.
3) Let an output map $h : X \to Y$ be $h(1) := a_1(1) = 0$, $h(2) = a_2(1) = -1$ and $h(3) = a_3(1) = 1$.
4) Since $S_l(-1)a_1 = a_2$, $S_l(-1)a_2 = a_1$ and $S_l(-1)a_3 = a_1$, $F(-1)$ is given by $(F(-1))(1) = 2$, $(F(-1))(2) = 1$ and $(F(-1))(3) = 1$.
5) Since $S_l(0)a_1 = a_1$, $S_l(0)a_2 = a_3$ and $S_l(0)a_3 = a_3$, $F(0)$ is given by $(F(0))(1) = 1$, $(F(0))(2) = 3$ and $(F(0))(3) = 3$.
The General Dynamical System that realizes the black-box can be expressed by the Figure 3.3.

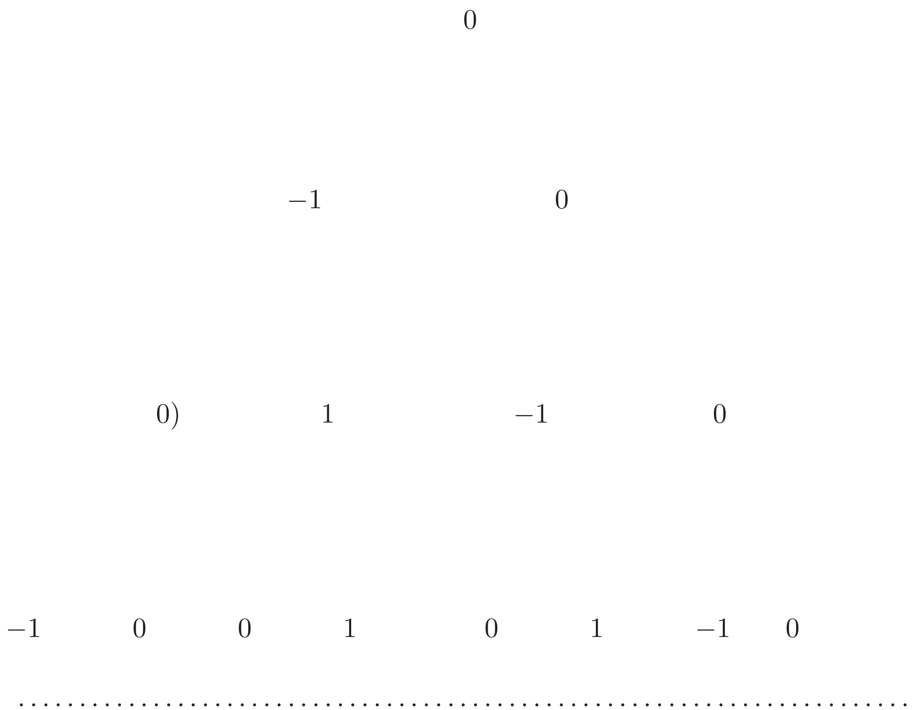
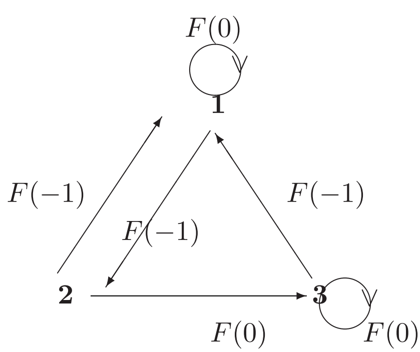


Fig. 3.2. The table of an input response map $a \in F(\Omega, Y)$, where $U = \{\alpha, \beta\}$



x	1	2	3
$h(x)$	0	-1	1

Fig. 3.3. A canonical General Dynamical System of Example (3.12)

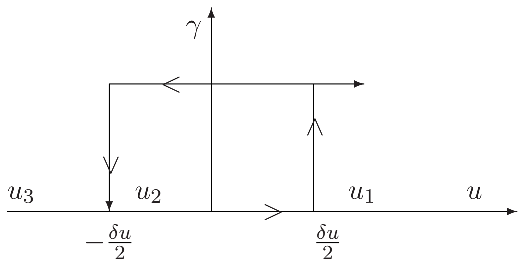


Fig. 3.4. A hysteresis characteristics of Example (3.13)

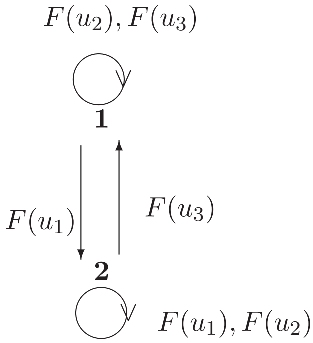
(3.13) Example
Let's consider a hysteresis characteristic (Figure 3.4), which is given by the following relation.

$$\gamma(t) := \begin{cases} 0 & u < -\frac{\delta u}{2} \\ 1 & \frac{\delta u}{2} < u \\ \gamma(t-1) & -\frac{\delta u}{2} \leq u \leq \frac{\delta u}{2} \end{cases}$$

Applying Theorem (3.11), we can obtain the General Dynamical System in Figure 3.5 that realizes the characteristics.
Where $u_1 > \frac{\delta u}{2}, -\frac{\delta u}{2} \leq u_2 \leq \frac{\delta u}{2}$ and $u_3 < -\frac{\delta u}{2}$.

3.3 Control Systems and Multi-experiments

Here we introduce a control system with an input mechanism and state structures for discrete-time systems, and we define its controllability and minimum control systems. The existence and uniqueness theorem for the minimum control is obtained. The control systems can be considered as automata with infinite state set, and these theorems will be applied to realization problems for dynamical systems. Lastly, as an application of control theory, a condition for a given black-box to be obtained by a single experiment is given.



x	1	2
$h(x)$	0	b

Fig. 3.5. A canonical General Dynamical System of Example (3.13)

(3.14) Definition
Let (X, F) be a U-action, T be a subset of X, then a triple $((X, F), T)$ is said to be a U-action with target. If there exists $\omega \in \Omega$ such that $\phi_F(\omega)x \in T$ for any $x \in X$, then a U-action with target $((X, F), T)$ is said to be controllable.

(3.15) Definition
Let $((X, F), x^0)$ be a U-action with an initial state and $((X, F), T)$ be a U-action with target. Then $C = ((X, F), x^0, T)$ is said to be a control system. If $C = ((X, F), x^0, T)$ is reachable and controllable, it is said to be trim. Let $|C| := \{\omega \in \Omega; \phi_F(\omega)x^0 \in T\}$, it is called a set of solutions.

Remark 1: If the set U of input values and the state set X is finite, then the control system $C = ((X, F), x^0, T)$ is the only automaton.
Remark 2: By correspondence given by Example (3.7), the control system $C = ((X, F), x^0, T)$ can be considered as the General Dynamical System $((X, F), x^0, \chi_T)$. Moreover, $a^{-1}(\mathit{1}) = |C|$ holds.

(3.16) Definition
Let $C_1 = ((X_1, F_1), x_1^0, T_1)$ and $C_2 = ((X_2, F_2), x_2^0, T_2)$ be control systems, if a map $T : X_1 \to X_2$ satisfies $Tx_1^0 = x_2^0$ and $T^{-1}(T_2) = T_1$, T is said to be a control system morphism $T : C_1 \to C_2$.

Remark: The definition of the control system morphism is equal to the dynamical system morphism corresponding to it.

(3.17) Proposition
Let $C_1 = ((X_1, F_1), x_1^0, T_1)$ and $C_2 = ((X_2, F_2), x_2^0, T_2)$ be control systems and $T : C_1 \to C_2$ be the control system morphism, then $|C_1| = |C_2|$ holds.

[proof] Let $\omega \in |C_1|$, i.e. $\phi_{F_1}(\omega)x_1^0 \in T_1$. $T\phi_{F_1}(\omega)x_1^0 = \phi_{F_2}(\omega)Tx_1^0 = \phi_{F_2}(\omega)x_2^0 \in T(T_1) \subseteq T_2$, hence $\omega \in |C_2|$ holds. Conversely, let $\omega \in |C_2|$, $\phi_{F_2}(\omega)x_2^0 = \phi_{F_2}(\omega)Tx_1^0 = T\phi_{F_1}(\omega)x_1^0 \in T_2$. Hence $\phi_{F_1}(\omega)x_1^0 \in T^{-1}(T_2) = T_1$.

(3.18) Proposition
Let $((X_1, F_1), T_1)$ and $((X_2, F_2), T_2)$ be U-actions with target, $T : (X_1, F_1) \to (X_2, F_2)$ be U-morphism and $T^{-1}(T_2) = T_1$. If $((X_2, F_2), T_2)$ is controllable, then $((X_1, F_1), T_1)$ is so. Conversely, if $((X_1, F_1), T_1)$ is controllable and T is surjective, then $((X_2, F_2), T_2)$ is controllable.

[proof] The first half: Let $x \in X_1$ be any element, then the assumption implies that there exist $x' \in X_2$ and $\omega \in \Omega$ such that $Tx = x'$ and $\phi_{F_2}(\omega)x' \in T_2$. $\phi_{F_2}(\omega)x' = \phi_{F_2}(\omega)Tx = T\phi_{F_1}(\omega)x$ holds. Hence, we obtain that $T-1T\phi_{F_1}(\omega)x = \phi_{F_1}(\omega)x \in T_1$. Secondly, we note the later half. There exist $x_1 \in X_1$ and $\bar{\omega} \in |omega$ such that $Tx_1 = x_2$ and $\phi_{F_1}(\bar{\omega})x_1 \in T_1$. $\phi_{F_2}(\bar{\omega})x_2 \in T_2$ holds because $TT_1 \ni T\phi_{F_1}(\bar{\omega})x_1 = \phi_{F_2}(\bar{\omega})Tx_1 = \phi_{F_2}(\bar{\omega})x_2$.

Let $C = ((X, F), x^0, T)$ be a control system, we can consider U-morphism $G : (\Omega, |) \to (X, F)$ by $G(u) = F(u)x^0u \in U$. Then $((\Omega, |), 1, G^{-1}(T))$ is said to be the input-control system.

By Proposition (3.18), we can obtain the following proposition easily:

(3.19) Proposition
A criterion for being trim of a control system is that the input-control system $((\Omega, |), 1, G^{-1}(T))$ is controllable.

(3.20) Definition
Let $C = ((X, F), x^0, T)$ be a control system. If the General Dynamical System $\sigma = ((X, F), x^0, \chi_T)$ corresponding to C is distinguishable, then σ is called a minimum control system.

Remark: If then set U of input values and a state set X are finite, the minimum control system C is a minimum automaton.
Hence, the minimum control system is an extension of automaton.
For a control system $C = ((X, F), x^0, T)$, $|C| = G^{-1}(T)$ holds. Where G is an input map : $(\Omega, |) \to (X, F)$ corresponding to the initial state x^0. Because of this and Proposition (3.19), we can obtain the following theorem easily.

(3.21) Theorem
Let $A \subseteq \Omega$. There exists a control system $C = ((X, F), x^0, T)$ which satisfies $|C| = A$ if and only if the input control system $((\Omega, |), \mathit{1}, A)$ is controllable.

(3.22) Theorem
Let $C_1 = ((X_1, F_1), x_1^0, T_1)$ and $C_2 = ((X_2, F_2), x_2^0, T_2)$ be minimum control systems with the same set of solutions. Then there exists a unique isomorphism $T : C_1 \to C_2$.

At the last of this chapter we discuss possibilities of multi-experiment as a control theory's application.
To determine an input response map that is a black-box B, we must measure the output by adding all input $\omega \in \Omega$ to the black-box B. However if we add an input $\omega \in \Omega$ into the input response map a (equivalently, B) then the black-box has changed into $S_l(\omega)a$. Therefore, infinite black-boxes B must be needed so as to determine a. This implies multi-experiment. A condition for the input response a to be determined with only one black-box is equal to the condition for the changed input response map $S_l(\omega)a$ to recover, namely it is equivalent to the controllable condition of $((S_l(\Omega)a, S_l), a, \{a\})$.

(3.23) Proposition
A given input response map a can be determined by experiment if and only if there exists $\bar{\omega} \in \Omega$ such that $S_l(\bar{\omega}|\omega)a = a$ for any $\omega \in \Omega$.

3.4 Historical Notes and Concluding Remarks

For General Dynamical Systems, Kalman defined more general and more complicated dynamical systems that include time-varying systems, but it is difficult to treat these dynamical systems mathematically.

Our General Dynamical Systems may be adequate because we obtained Realization Theorem (3.6). We have established a basis of the general realization theory and the control theory for all systems which include non-linear, time-varying and infinite dimensional systems. By using the uniqueness of the concatenation monoid, we have shown that state structures of our state space become simple, namely Ω-module can be completely characterized by U-action (see Appendix 3.5.A). This means that a state transition equation is described by a difference equation. These are the same as the automata's theory of Eilenberg.

In Theorem (3.4), we gave two canonical realizations for an arbitrary input response map. The quotient set of them may be derived from Nerode equivalence [1958], and the set may be closed to right congruence in Eilenberg [1974]. Note that Nerode proposed the so-called nerode equivalence for automata with linear input/output maps. This concept was used in the finite dimensional linear systems by Kalman [1965] and in the infinite dimensional linear systems by Balarkrishnan [1965], [1967] and Matsuo [1968], [1969]. The image set of the canonical General Dynamical System 2) in Theorem (3.4) may have close relations to the reduced form in automata to be derived from subset construction, which is related to $F(\Omega, Y)$. For example, see Eilenberg [1974]. This concept was used in the reachable set of $K[[z^{-1}]]$ by Kalman, Falb and Arbib [1969].

Zeiger [1967] proved the uniqueness theorem for discrete-time finite dimensional linear systems. On the other hand, for automata theory, Eilenberg [1974] discussed the morphism of state transitions by using T_{12}^{max} used in Appendix 3.5.D. Matsuo [1975] proved the uniqueness theorem for continuous-time infinite dimensional linear systems by introducing also T_{12}^{min} used in Appendix 3.5.D. By extending the idea, Matsuo [1981] obtained the uniqueness theorem for continuous-time General Dynamical Systems. Isidori [1975] obtained the uniqueness theorem for finite dimensional inhomogeneous bilinear systems by using Zeiger's Lemma.

In this chapter we have obtained the uniqueness theorem by using Matsuo's idea. We have shown that an automaton is a special dynamical system of our General Dynamical Systems. We have introduced the minimal control systems which are extensions of minimal automaton, and we have obtained the existence theorem and uniqueness theorem for them. Also we have given the condition for a given black-box to be obtained by multi-experiment, which is an application of the minimal control systems.

3.5 Appendix

Matsuo solved the realization problem of any input/output map of continuous-time system by introducing the General Dynamical System of continuous-time system. We also solved the realization problem of any input/ output map of discrete-time system by Realization Theorem (3.6). We will discuss relations between ones of continuous-time system and ones of discrete-time system. Then we will clarify the special features of discrete-time system by rewriting in the suitable form of discrete-time. These in discrete-time system will arise from the uniqueness of the concatenation monoid and a fact that it is a free monoid over the input set U.

3.5.A U-Action

Here we introduce the Ω-module introduced by Matsuo [1981].

(3-A.1) Definition

Let X be any set and ϕ be a monoid morphism: $\Omega \to F(X)$. Where $\phi(\omega_2|\omega_1) = \phi(\omega_2)\phi(\omega_1)$ and $\phi(\mathit{1}) = I$ (Identity map on X) hold for $\omega_2, \omega_1 \in \Omega$. Then a pair (X, ϕ) is said to be a Ω-module, X is called a state set and ϕ is called a transition map.

Remark: Let (X, ϕ) be a Ω-module. $\phi(\omega) : X \to X$ can be considered as a transformation from a state $x(0)$ at time 0 to a state $x(n)$ at time n for $n := |\omega|$ and $\omega \in \Omega$. Hence the Ω-module (X, ϕ) implies the transition equation $x(n) = \phi(\omega)x(0)$ and the equation satisfies the following two properties.
1: consistency; $\phi(\mathit{1})x = x$ for any $x \in X$.
2: composition property; $\phi(\omega_2|\omega_1)x = \omega(\omega_2)\omega(\omega_1)x$ for any ω_2, ω_1 and $x \in X$. Therefore, (X, ϕ) can express a general transition equation by Matsuo [1981].

Next, we show the following lemma needed later.

(3-A.2) Lemma

Let M be any monoid and a map i be i: $U \to \Omega$; $u \mapsto u$, then there exists uniquely a monoid morphism $\hat{f} : \Omega \to M$ such that $f = \hat{f} \cdot i$ for any map $f : \Omega \to M$. This correspondence is bijective.

[proof] See Chevalley [1956].

(3-A.3) Proposition
Let X be a set and $Mor(U, F(X))$ be a set of any transition map from U to $F(X)$. Let $F(\Omega, F(X))$ be a set of any function from Ω to $F(X)$. Then $Mor(U, F(X)) = F(\Omega, F(X))$ holds.

[proof] Let M in Lemma (3-A.2) be $F(X)$. Then this proposition can be obtained.

Remark: Let X be a state set and $F \in F(U, F(X))$. Then note that a pair (X, F) is a U-action (see Definition (3.1)). The proposition insists that there exists a bijective correspondence between a U-action and an Ω-module. Let (X, ϕ) be Ω-module corresponding to (X, F), then (X, ϕ) may be written by (X, F). Note that the U-action (X, F) implies the following state difference equation $x(n+1) = F(u)x(n)$ for any $n \in N$. This is the first special feature of discrete-time system.

(3-A.4) Example
$(\Omega, |)$ is an Ω-module by $\omega| : \Omega \to \Omega; \bar{\omega} \mapsto \omega|\bar{\omega}$.
The concatenation $|$ which is the monoid morphism $:\Omega \to F(\Omega)$ shall be regarded as a map $| : U \to F(\Omega)$, hence $(\Omega, |)$ will be a U-action.

(3-A.5) Example
Let $a \in F(\Omega, Y)$ be any input response map and let S_l be defined by $S_l(\omega)a : \Omega \to Y\ ;\ \bar{\omega} \mapsto a(\bar{\omega}|\omega)$. Then $S_l(\omega)$ is a monoid morphism $: \Omega \to F(\Omega, Y)$. Hence, the pair $(F(\Omega, Y), S_l)$ is an Ω-module. Note that S_l of Ω-module $(F(\Omega, Y), S_l)$ is used in the same way as S_l of U-action $(F(\Omega, Y), S_l)$ in Example (3.3).

In Matsuo [1981], the Ω-morphism $T : (X_1, \phi_1) \to (X_2, \phi_2)$ has been defined by a map $T : X_1 \to X_2$ such that $T\phi_1(\omega) = \phi_2(\omega)T$ for any $\omega \in \Omega$.
Here we will define the following:

(3-A.6) Definition
Let (X_1, F_1) and (X_2, F_2) be U-actions. A map $T : X_1 \to X_2$ defined by $TF_1(u) = F_2(u)T$ for any $u \in U$ is said to be a U-morphism $T : (X_1, F_1) \to (X_2, F_2)$.

(3-A.7) Proposition
Let (X_1, ϕ_1) and (X_2, ϕ_2) be Ω-modules. And let (X_1, F_1) and (X_2, F_2) be U-actions which correspond to them respectively.
Then a map T is the Ω-morphism $T : (X_1, \phi_1) \to (X_2, \phi_2)$ if and only if T is the U-morphism $T : (X_1, F_1) \to (X_2, F_2)$.

[proof] Necessity: Let $T\phi_1(\omega) = \phi_2(\omega)T$ for any $\omega \in \Omega$. Since $u \in U$ implies $u \in \Omega$, $TF_1(u) = T\phi_1(u) = \phi_2(u)T = F_2(u)T$ hold.
Sufficiency: There exist $u_i \in U$ $(1 \leq i \leq n)$ such that $\omega = u_n|u_{n-1}|\cdots|u_1$ for any $\omega \in \Omega$. Then $T\phi_1(\omega) = TF_1(u_n)F_1(u_{n-1})\cdots F_1(u_1) =$
$F_2(u_n)TF_1(u_{n-1})\cdots F_1(u_1) = \cdots = F_2(u_n)F_2(u_{n-1})F_2(u_1)T = \phi_2(\omega)T$ hold.

Remark: The U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ means that the state-difference equation in X_1 is preserved into the state-difference equation in X_2. Proposition (3.5) insists that Ω-morphism (equivalently, preservation of state-transition equation) is completely equivalent to U-morphism (preservation of state-difference equation). This is the second special feature in discrete-time case.

3.5.B Pointed U-Actions

Here, we will introduce a U-action that has a structure of input.

(3-B.1) Definition
Let (X, F) be a U-action and an initial state $x^0 \in X$. Then a triple $((X, F), x^0)$ is said to be a pointed U-action. A pointed U-action $((X, F), x^0)$ means the following equations:

$$\begin{cases} x(t+1) = F(\omega(t+1))x(t) \\ x(0) = x^0 \end{cases}$$

for any $t \in N$, where $x(t) \in X$.
Let (X, ϕ_F) be the Ω-module corresponding to the U-action (X, F). If $((X, \phi_F), x^0)$ is reachable, namely there exists $\omega \in \Omega$ such that $x = \phi_F(\omega)x^0$ for any $x \in X$, then $((X, F), x^0)$ is said to be a reachable pointed U-action.

(3-B.2) Example
Let $(\Omega, *|)$ be the U-action discussed in Example (3.3) and a unit element

$\mathit{1}$ be an initial state. Then $((\Omega, *|), \mathit{1})$ is a reachable pointed U-action. Let $(F(\Omega, Y), S_l)$ be the U-action discussed in Example (3.2) and an input response map $a \in F(\Omega, Y)$ be an initial state. Then $((F(\Omega, Y), S_l), a)$ is a pointed U-action.

Let (X, F) be a U-action and G be U-morphism : $(\Omega, *|) \to (X, F)$, then $G : (\Omega, *|) \to (X, F)$ is said to be an input map and $((X, F), G)$ is said to be a U-action with an input map. We assume that $UMor(\Omega, X)$ is a set of any input map $G : (\Omega, *|) \to (X, F)$.

(3-B.3) Proposition

Let (X, F) be a U-action. A map $1 : UMor(\Omega, X) \to X; G \mapsto G(\mathit{1})$ is a bijection. Namely, there is a bijective correspondence between the U-action with an input map $((X, F), G)$ and the pointed U-action $((X, F), x^0)$.
Where $G(\mathit{1}) = x^0$ and $G(u) = F(u)x^0$ hold.

[proof] See Matsuo [1977] or [1981].

(3-B.4) Corollary

For any input response map $a \in F(\Omega, Y)$, there exists a uniquely determined U-morphism $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$ such that $A(\mathit{1}) = a$.
This correspondence is bijective.
Where $A(\omega(n)|\omega(n-1)|\cdots|\omega(1)) = S_l(\omega(n))S_l(\omega(n-1))|\cdots|S_l(\omega(1))a$.

[proof] Let (X, F) in Proposition (3-B.3) be $(F(\Omega, Y), S_l)$ in Example (3.2). Then this corollary is easily obtained.

Corollary (3-B.4) implies that an input response map $a \in F(\Omega, Y)$ can be expressed by the U-morphism $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$.
This A is said to be a (sophisticated) input/output map.

(3-B.5) Proposition

A pointed U-action $((X, F), x^0)$ is reachable if and only if G in the U-action with an input map $((X, F), G)$ corresponding to it is surjective.

[proof] This is obtained by definition of reachability.

3.5.C U-Actions with a Readout Map

Here, we will introduce U-actions with a structure of output, which are said to be U-actions with a readout map.

(3-C.1) Definition
Let (X, F) be a U-action and h be any map : $X \to Y$. Then a triple $((X, F), h)$ is said to be a U-action with a readout map. Let (X, ϕ) be the Ω-module corresponding to the U-action (X, F).
If $((X, F), h)$ is distinguishable, namely $h\phi_F(\omega)x_1 = h\phi_F(\omega)x_2$ for any $\omega \in \Omega$ implies $x_1 = x_2$, then $((X, F), h)$ is said to be a distinguishable U-action with a readout map.

(3-C.2) Example
Let a be an input response map. Then $((\Omega, *|), a)$ is a U-action with a readout map. Moreover, for a map $1 : F(\Omega, Y) \to Y; a \mapsto a(\mathit{1})$, $((F(\Omega, Y), S_l), 1)$ is a distinguishable U-action with a readout map.

Let (X, F) be a U-action and H be U-morphism : $(X, F) \to (F(\Omega, Y), S_l)$. then $H : (X, F) \to (F(\Omega, Y), S_l)$ is said to be an observation map and $((X, F), H)$ is said to be a U-action with an observation map. We assume that $UMor(X, F(\Omega, Y))$ is a set of any observation maps $H : (X, F) \to (F(\Omega, Y), S_l)$.

(3-C.3) Proposition
Let (X, F) be a U-action. A map $1 : UMor(X, F(\Omega, Y)) \to F(X, Y); H \mapsto 1 \cdot H := h$ is a bijection. Namely, there is a bijective correspondence between the U-action with an observation map $((X, F), H)$ and the U-action with a readout map $((X, F), h)$, where $Hx(\omega) = h \cdot \phi_F(\omega)x$ hold for any $\omega \in \Omega$ and $x \in X$.

[proof] See Matsuo [1977] or [1981].

Remark: Let (X, F) in Proposition (3-C.3) be $(\Omega, *|)$, then there uniquely corresponds the observation map $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$ (note that it is a sophisticated input/output map) to an output map $a \in F(\Omega, Y)$ (note that it is an input response map) by $A(\omega)(\bar{\omega}) = a(\bar{\omega}|\omega)$ for any $\bar{\omega}, \omega \in \Omega$. (See also Corollary (3-B.4).)

(3-C.4) Proposition
A U-action with a readout map $((X, F), h)$ is distinguishable if and only if H in the U-action with an observation map $((X, F), H)$ corresponding to it is injective.

[proof] This is obtained by definition of distinguishable.

3.5.D General Dynamical Systems

General Dynamical Systems are composed of a pointed U-action and a U-action with a readout map. Therefore, based on the results of sections 3-B and 3-C, we will discuss details of General Dynamical Systems.

(3-D.1) Definition
Let $((X, F), x^0)$ be a pointed U-action and $((X, F), h)$ be a U-action with a readout map. Then a collection $\sigma = ((X, F), x^0, h)$ is said to be a (naive) General Dynamical System.
Let $((X, F), G)$ be a U-action with an input map and $((X, F), H)$ be a U-action with an observation map, then $\Sigma = ((X, F), G, H)$ is said to be a sophisticated General Dynamical System. For a naive General Dynamical System $\sigma = ((X, F), x^0, h)$, an input response map $a_\sigma : \Omega \to Y; \omega \mapsto h \cdot F(\omega(|\omega|))F(\omega(|\omega| - 1)) \cdots F(\omega(1))x^0$ is said to be the behaviour of σ.

For a sophisticated General Dynamical System $\Sigma = ((X, F), G, H)$, an input/output map $A = H \cdot G : (\Omega, *|) \to (F(\Omega, Y), S_l)$ is said to be a behavior of Σ. A naive General Dynamical System $\sigma = ((X, F), x^0, h)$ [A sophisticated General Dynamical System $\Sigma = ((X, F), G, H)$] is called canonical if $((X, F), x^0)$ is reachable [G is surjective] and $((X, F), h)$ is distinguishable [H is injective].

(3-D.2) Proposition
Let $N.G.D.S.$ be a category of naive General Dynamical Systems, and $S.G.D.S.$ be a category of sophisticated General Dynamical Systems. Then $N.G.D.S. = S.G.D.S.$ holds.

[proof] This can be obtained easily by Propositions (3-B.3) and (3-C.3).

Remark: Let $\sigma = ((X, F), x^0, h)$ be a naive General Dynamical System and $\Sigma = ((X, F), G, H)$ be a sophisticated General Dynamical System corre-

sponding to σ [see Propositions (3-B.3) and (3-C.3)], then the following relation holds.
$(A_\Sigma(\mathit{1}))(\omega) = (H \cdot G(\mathit{1}))(\omega) = (H \cdot x^0)(\omega) = h\phi_F(\omega)x^0 = a_\sigma(\omega)$, any $\omega \in \Omega$ between a_σ and A_Σ. Namely A_Σ is the input/output map corresponding to the input response map a_σ. And the converse relation holds.

Let $\sigma_1 = ((X_1, F_1), x_1^0, h_1)$ [$\Sigma_1 = ((X_1, F_1), G_1, H_1)$] and $\sigma_2 = ((X_2, F_2), x_2^0, h_2)$ [$\Sigma_2 = ((X_2, F_2), G_2, H_2)$] be General Dynamical Systems. If a map $T : X_1 \to X_2$ satisfies $Tx_1^0 = x_2^0$ [$TG_1 = G_2$] and $h_1 = h_2T$ [$H_1 = H_2T$], then T is said to be a dynamical system morphism $T : \sigma_1 \to \sigma_2$ [$T : \Sigma_1 \to \Sigma_2$]. If the T is bijective then T is said to be isomorphism.

(3-D.3) Lemma
Let $\sigma_1 = ((X_1, F_1), x_1^0, h_1)$ and $\sigma_2 = ((X_2, F_2), x_2^0, h_2)$ be General Dynamical Systems, and T be a dynamical system morphism $T : \sigma_1 \to \sigma_2$. Then $a_{\sigma_1} = a_{\sigma_2}$ holds.

[proof] Let ϕ_{F_1} and ϕ_{F_2} be transition morphism corresponding to F_1 and F_2 respectively. Then $a_{\sigma_1}(\omega) = h_1\phi_{F_1}(\omega)x_1^0 = h_2T\phi_{F_1}(\omega)x_1^0 = h_2\phi_{F_2}(\omega)Tx_1^0 = h2\phi_{F_2}(\omega)x_2^0 = a_{\sigma_2}(\omega)$ holds.

(3-D.4) Example
For any input response a, the reachable General Dynamical System $((\Omega, *|), \mathit{1}, a)$ and the distinguishable General Dynamical System $((F(\Omega, Y), S_l), a, 1)$ are ones that realize a. (See Examples (3.2) and (3.3))
For $A \in UMor(\Omega, F(\Omega, Y))$, the reachable sophisticated General Dynamical System $\Sigma = ((\Omega, *|), I, A)$ and distinguishable General Dynamical System $\Sigma = ((F(\Omega, Y), S_l), A, I)$ are ones which realize A.
Where I are the Identity map on Ω and $F(\Omega, Y)$ respectively.

(3-D.5) Sub U-actions
Let (X, F) be a U-action and $Z \subseteq X$ be an invariant sub set under F, i.e., any $z \in Z$ implies $F(u)z \in Z$ for $u \in U$. Setting $F_Z(u) := F(u)|_Z$ (restriction of $F(u)$ to Z), (Z, F_Z) is a U-action, it is said to be a sub U-action.

(3-D.6) Quotient U-actions
Let (X, F) be a U-action and an equivalence relation R in X ($x_1Rx_2 \iff h(x_1) = h(x_2)$ by a map $h : X \to Y$) consists with F, i.e., x_1Rx_2 implies

$F(u)x_1 R F(u)x_2$ for any $u \in U$. Then we can introduce a map $\hat{F}(u) : X/_R \to X/_R; [x] \mapsto [F(u)x]$, we obtain a quotient U-action $(X/_R, \hat{F})$.

The realization problem of General Dynamical Systems stated before was the following: For any input response map $a \in F(\Omega, Y)$, find a General Dynamical System σ that satisfies $a_\sigma = a$ in the category of General Dynamical Systems. This problem can be rewritten equivalently as the following problem by Proposition (3-D.2).

Realization problem in sophisticated sense:
[For any input/output map $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$, find a sophisticated General Dynamical System $\Sigma = ((X, F), G, H)$ which satisfies $A_\Sigma = A$ within the category of sophisticated General Dynamical Systems.]

In order to solve the realization problem, we discuss the problem for sophisticated General Dynamical Systems subsequently. According to Example (3-D.4), there exist many General Dynamical Systems that realize an input/output map $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$, but there is no relation between them. Therefore, we consider only the canonical General Dynamical Systems that realize A, then we insist that there exist always the canonical General Dynamical Systems that realize it, and they are isomorphic.

(3-D.7) Proposition
A U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ can be naturally decomposed into the following:
$X_1 \overset{\pi}{\to} X_1/_T \overset{T^b}{\to} \text{im } T \overset{j}{\to} X_2$
Where $X_1/_T$ is the quotient set induced by the map T, π is the natural surjection, T^b is the bijection associated with T and j is the natural injection. Moreover, they are also U-morphisms respectively.

(3-D.8) Theorem
For any input/output map $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$, there exist the two following canonical General Dynamical Systems that realize it.
1) $\Sigma_q = ((\Omega/_A, \hat{*}|), \pi, A^i)$
Where π is the natural surjection $: \Omega \to \Omega/_A$, A^i is given by $A^i = j \cdot A^b$ for natural bijection A^b associated with A and the natural injection $j : ImA \to F(\Omega, Y)$.
2) $\Sigma_s = ((ImA, S_l), A, I)$
Where $A^b = A^s \cdot \pi$.

To obtain the uniqueness part of realization theorem, we will introduce a morphism from $\Sigma_1 = ((X_1, F_1), G_1, H_1)$ to $\Sigma_2 = ((X_2, F_2), G_2, H_2)$ as the following:
$Mor(\Sigma_1, \Sigma_2) := \{$a relation $T_{12} : X_1 \to X_2; GrT_{12}^{min} \subseteq GrT_{12} \subseteq GrT_{12}^{max}\}$.
Where GrT_{12}^{min} is the graph of the relation $T_{12}^{min} := G_2 \cdot G_1^{-1}$, GrT_{12} is the graph of T_{12} and GrT_{12}^{max} is the graph of $T_{12}^{max} := H_2^{-1} \cdot H_1$. Then we can obtain the following lemmas in the same manner as in reference Matsuo [1981].

(3-D.9) Lemma
$A_{\Sigma_1} = A_{\Sigma_2}$ holds if and only if $Mor(\Sigma_1, \Sigma_2) \neq \phi$.

(3-D.10) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold.
1) If G_1 in Σ_1 is surjective then dom $T_{12}^{min} = X_1$ holds.
Where dom T_{12}^{min} is the domain of T_{12}^{min}.
2) If H_2 in Σ_2 is injective then T_{12}^{max} is a partial function: $X_1 \to X_2$.

(3-D.11) Product U-actions
Let (X_1, F_1) and (X_2, F_2) be U-actions. If we introduce a map $F_1 \times F_2$: $U \to F(X_1 \times X_2)$; $u \mapsto [; (x_1, x_2) \mapsto (F_1(u)x_1, F_2(u)x_2)]$ on a product set $X_1 \times X_2$, then $(X_1 \times X_2, F_1 \times F_2)$ is a U-action. It is said to be a product U-action.

(3-D.12) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold. Then Gr T_{12}^{max} is a sub product U-action of (X_1, F_1) and (X_2, F_2).

(3-D.13) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, G_1 in Σ_1 be surjective and H_2 in Σ_2 be injective, then $T_{12}^{min} = T_{12}^{max}$ holds. Let $T_{12} := T_{12}^{min}$, then T_{12} is a system morphism : $\Sigma_1 \to \Sigma_2$, i.e., T_{12} is a U-morphism : $(X_1, F_1) \to (X_2, F_2)$ satisfies $T_{12} \cdot G_1 = G_2$ and $H_1 = H_2 \cdot T_{12}$.

(3-D.14) Theorem
Let $\Sigma_1 = ((X_1, F_1), G_1, H_1)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2)$ be canonical realizations of an input/output map $A : (\Omega, *|) \to (F(\Omega, Y), S_l)$, then there exists a uniquely isomorphic system morphism $T_{12} : \Sigma_1 \to \Sigma_2$.

Remark: This theorem and Proposition (3-D.2) say that the uniqueness part of Theorem (3.6) is proved.

(3-D.15) Proof of Theorem (3.11)
Let a map $T : S_l(\Omega)a \to X$ be ; $a_i \mapsto i$. By step 3), $1 = h \cdot T$ holds. By step 4) and 5), $T \cdot S_l = FT$ holds. Therefore, T is a dynamical system morphism : $((S_l(\Omega)a, S_l), a, 1) \to ((X, F), 1, h)$ and a bijective map : $S_l(\Omega)a \to X$. By Lemma (3-D.3), the behavior of $((S_l(\Omega)a, S_l), a, 1)$ is equal to one of $((X, F), 1, h)$.

4 Linear Representation Systems

Let the set of output's values Y be a linear space over the field K. Linear Representation Systems are presented with the following main theorem. The main theorem says that for any causal input/output map (equivalently, any input response map), there exist at least two canonical (quasi-reachable and distinguishable) Linear Representation Systems which realize (faithfully describe) it and any two canonical Linear Representation Systems with the same behavior are isomorphic.

Firstly, their realization theory is stated.

Secondly, details of finite dimensional Linear Representation Systems are investigated. We give a criterion for the canonicality of finite dimensional Linear Representation Systems. We give representation theorems of isomorphic classes for canonical Linear Representation Systems. We give criterions for the behavior of finite dimensional Linear Representation Systems. In adition, a procedure to obtain a canonical Linear Representation System is given.

Thirdly, their partial realization is discussed according to the above results. Existence of minimum partial realization is trivially presented. It rarely happens for minimum partial realizations to be unique up to isomorphism. To solve the uniqueness problem, we introduce the notion of natural partial realizations. The main results for the partial realization are the following:

1) A necessary and sufficient condition for the existence of the natural partial realizations is given by the rank condition of finite sized Hankel matrix.

2) The existence condition of natural partial realization is equivalent to the uniqueness condition of minimum partial realizations.

3) An algorithm to obtain a natural partial realization from a given partial input response map is given.

We can easily understand that the above results of our systems are the same as ones obtained in linear system theory.

4.1 Realization Theory of Linear Representation Systems

(4.1) Definition

A system given by the following equations is written as a collection $\sigma = ((X, F), x^0, h)$ and it is said to be a Linear Representation System.

T. Matsuo, Y. Hasegawa: Realization Theory of Discrete-Time Dynamical Systems, LNCIS 296, pp. 38–78, 2003.

$$\begin{cases} x(t+1) &= F((t+1))x(t) \\ x(0) &= x^0 \\ \gamma(t) &= hx(t) \end{cases}$$

for any $t \in N$, $x(t) \in X$, $\gamma(t) \in Y$.
Where X is a linear space over the field K, F is a linear operator on X, a initial state $x^0 \in X$ and $h : X \to Y$ is a linear operator.
The input response map $a_\sigma : \Omega \to Y; \omega \mapsto h\phi_F(\omega)x^0$ is said to be the behavior of σ. For an input response map $a \in F(\Omega, Y)$, σ which satisfies $a_\sigma = a$ is called a realization of a.
A Linear Representation System σ is said to be quasi-reachable if the linear hull of the reachable set $\{\phi_F(\omega)x^0; \omega \in \Omega\}$ is equal to X and a Linear Representation System σ is called distinguishable if $h\phi_F(\omega)x_1 = h\phi_F(\omega)x_2$ for any $\omega \in \Omega$ implies $x_1 = x_2$.
A Linear Representation System σ is called canonical if σ is quasi-reachable and distinguishable.

Remark 1: The $x(t)$ in the system equation of σ is the state that produces output values of a_σ at the time t, namely the state $x(t)$ and linear operator $h : X \to Y$ generate the output value $a_\sigma(t)$ at the time t.
Remark 2: It is meant for σ to be a faithful model for the input response map a that σ realizes a.
Remark 3: Notice that a canonical Linear Representation System $\sigma = ((X, F), x^0, h)$ is a system that has the most reduced state space X among systems that have the behavior a_σ (see Definition (4-A.23), Proposition (4-A.27), (4-A.28), Definition (4-A.29), Propositions (4-A.32), (4-A.33), Definition (4-A.34), Proposition (4-A.36) and Corollary (4-A.20) in Appendix 4.5.)

(4.2) Example
$A(\Omega) := \{\lambda = \sum_{\omega} \lambda(\omega)e_\omega$ (finite sum) $\}$.
Where $\omega = \bar{\omega}$ implies $e_\omega(\bar{\omega}) = 1$, and $\omega \neq \bar{\omega}$ implies $e_\omega(\bar{\omega}) = 0$. Let S_r be a map $: U \to L(A(\Omega)); u \mapsto S_r(u)[\lambda \mapsto \sum_{\omega} \lambda(\omega)e_{u|\omega}$, an initial state be e_1 and a linear output map be $a : A(\Omega) \to Y; \lambda \mapsto a(\lambda) = \sum \omega\lambda(\omega)a(\omega)$. Then a collection $((A(\Omega), S_r), e_1, a)$ is a quasi-reachable Linear Representation System that realizes a. (For a linear map $: A(\Omega) \to Y$, see Proposition (4-A.22))

Let $F(\Omega, Y)$ be a set of any input response maps, let $S_l : U \to L(A(\Omega)); u \mapsto S_l(u)[a \mapsto [\omega \mapsto a(\omega|u)]]$. Let a linear output map be $1 : F(\Omega, Y) \to Y; a \mapsto a(1)$. Then a collection $(F(\Omega, Y), S_l), a, 1)$ is a distinguishable Linear Representation System that realizes a.

Remark: Note that the linear output map $a : A(\Omega) \to Y$ is introduced by the fact $F(\Omega, Y) = L(A(\Omega), Y)$ (See Proposition (4-A.22)).

(4.3) Theorem
The following two Linear Representation Systems are canonical realizations of any input response map $a \in F(\Omega, Y)$.

1) $((A(\Omega)/_a, \hat{S}_r), [e_1], \hat{a})$.
Where $A(\Omega)/_a$ is a quotient space obtained by equivalence relation $\sum_{\omega} \lambda(\omega) e_\omega = \sum_{\bar{\omega}} \lambda(\bar{\omega}) e_{\bar{\omega}} \iff \sum_{\omega} \lambda(\omega) a(\omega) = \sum_{\bar{\omega}} \lambda(\bar{\omega}) a(\bar{\omega})$. $\hat{S}_r$ is given by a map $: U \to L(A(\Omega)/_a); u \mapsto \hat{S}_r(u)[\lambda \mapsto \sum_{\omega} \lambda(\omega)[e_{u|\omega}]$, and $\hat{a}$ is given by $\hat{a} : A(\Omega)/_a \to Y; [\lambda] \mapsto \hat{a}([\lambda]) = \sum_{\omega} \lambda(\omega) a(\omega)]$.

2) $((\ll S_l(\Omega)a \gg), S_l), a, 1)$.
Where $\ll S_l(\Omega)a \gg$ is the smallest linear space which contains $S_l(\Omega)a := \{S_l(\omega)a; \omega \in \Omega\}$.

[proof] See Remark 2 in Proposition (4-A.27) (or (4-A.32)). Also see Propositions (4-A.28), (4-A.33), (4-A.36) and Corollary (4-A.38).

(4.4) Definition
Let $\sigma_1 = ((X_1, F_1), x_1^0, h_1)$ and $\sigma_2 = ((X_2, F_2), x_2^0, h_2)$ be Linear Representation Systems, then a linear operator $T : X_1 \to X_2$ is said to be a Linear Representation System morphism $T : \sigma_1 \to \sigma_2$ if T satisfies $TF_1(u) = F_2(u)T$ for any $u \in U$, $Tx_1^0 = x_2^0$ and $h_1 = h_2 T$. If $T : X_1 \to X_2$ is bijective, then $T : \sigma_1 \to \sigma_2$ is said to be an isomorphism.

(4.5) Realization Theorem of Linear Representation Systems
For any input response map $a \in F(\Omega, Y)$, there exist at least two canonical Linear Representation Systems which realize a. (Existence part).
Let σ_1 and σ_2 be any two canonical Linear Representation Systems that realize $a \in F(\Omega, Y)$, then there exists an isomorphism $T : \sigma_1 \to \sigma_2$. (Uniqueness part).

[proof] The former part is the same as Theorem (4.3). The latter part is obtained by Remark of Lemma (4-A.42) in Appendix 4.5.

4.2 Finite Dimensional Linear Representation Systems

Based on the realization theory (4.5), we study structures of finite-dimensional Linear Representation Systems in this section. To obtain concrete and meaningful results, we assume that the set U of input values is finite; i.e., $U := \{u_i; 1 \leq i \leq m \text{ for some } m \in N\}$. This assumption implies that the difference morphism F of a Linear Representation System $\sigma = ((X, F, x^0, h)$ is completely determined by the finite matrices $\{F(u_i); 1 \leq i \leq m\}$. But it will be presented that the assumption is not so special. Main results can be stated in the following four steps:

Firstly, we present conditions when finite dimensional Linear Representation System is canonical.

Secondly, we obtain the representation theorem for finite dimensional canonical Linear Representation Systems, namely, we show two standard systems as a representative in their equivalence classes. One is the quasi-reachable standard system, and the other is the distinguishable standard system.

Thirdly, we give two criterions for the behavior of finite dimensional Linear Representation Systems. One is the rank condition of infinite Hankel matrix, and the other is the application of Kleene's Theorem obtained in automata theory.

Lastly, we give a procedure to obtain the quasi-reachable standard system which realizes a given input response map.

We will prove the above matters in Appendix 4.5.B.

(4.6) Corollary
Let T be a Linear Representation System morphism $T : \sigma_1 \to \sigma_2$, then $a_{\sigma_1} = a_{\sigma_2}$ holds.

[proof] This is a direct calculation by the definition of the behavior and Linear Representation System morphism.

There is a fact about finite dimensional linear spaces that an n-dimensional linear space over the field K is isomorphic to K^n and $L(K^n, K^m)$ is isomorphic to $K^{m\times n}$ (See Halmos [1958]). Therefore, without loss of generality, we can consider n-dimensional Linear Representation System as $\sigma = ((K^n, F), x^0, h)$, where F is a map : $U \to K^{n\times n}$, $x^0 \in K^n$ and $h \in K^{p\times n}$. Now we will show that the assumption of finiteness for input value's set U is not so special.

(4.7) $U = \{u_1, u_2\}$

In this case, a Linear Representation System $\sigma = ((K^n, F), x^0, h)$ can be completely determined by $\{F(u_i); u_i \in U \text{ for } i = 1, 2\}$.

If on-off inputs are applied to a black-box, any non-linear system can be treated in this case.

Moreover, if an optimal solution is a bang-bang control, when a controlled object is in the optimal controlled condition, then it can be treated in this case.

(4.8) Cases where U is a convex set in R^m

Let the set U be a convex set in R^m and a set V of the extreme points be a finite set $\{u_j; 1 \le j \le m\}$. Let F in $\sigma = ((K^n, F), x^0, h)$ be a linear operator : $U \to K^{n\times n}$, i.e. $F(\sum_{i=1}^m \alpha_i \mathbf{e_i}) = \sum_{i=1}^m \alpha_i F(u_i)$, $\sum_{i=1}^m \alpha_i = 1$. Then the Linear Representation System $\sigma = ((K^n, F), x^0, h)$ can be rewritten as a Linear Representation System $\tilde{\sigma} = ((K^n, \tilde{F}), x^0, h)$.

Where $\tilde{F} : V \to K^{n\times n}$ is given by $\tilde{F}(u_i) = F(u_i)$ for any $u_i \in V$.

Note that the quasi-reachability of σ is equivalent to the quasi-reachability of $\tilde{\sigma}$.

(4.9) $U = R^m$

Let $K = R$ and $V = \{0, \mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_m}\}$ for basis $\mathbf{e_i}$ in $R^m (1 \le i \le m)$. Let F in $\sigma = ((R^n, F), x^0, h)$ be an affine operator : $U \to R^{n\times n}$, i.e. $F(\sum_{i=1}^m \alpha_i \mathbf{e_i}) = A + (\sum_{i=1}^m \alpha_i \tilde{N}_i)$, A, $\tilde{N}_i \in R^{n\times n}$, $i \in N$. Then the Linear Representation System $\sigma = ((R^n, F), x^0, h)$ can be rewritten as a Linear Representation System $\tilde{\sigma} = ((R^n, \tilde{F}), x^0, h)$. Where $\tilde{F} : V \to K^{n\times n}$ is given by $F(0) = A$, $F(\mathbf{e_i}) = A + \tilde{N}_i$ for any $i (1 \le j \le m)$. Note that this $\tilde{\sigma}$ is a homogeneous bilinear system investigated by Tarn & Nonoyama [1976].

Note that the quasi-reachability of σ is equivalent to the quasi-reachability of $\tilde{\sigma}$.

(4.10) $K - U$-Automata

Extending an automaton, Eilenberg defined $K - U$-automaton $A = (Q, E, \underline{x}^0, \underline{h})$ which is a sort of deterministic automaton by the

following four conditions. Though Eilenberg discussed in the case where K is a semi-ring, here let K be the field.
1) U is an alphabet of finite elements (m elements).
2) Q is a state set of finite elements (n elements).
3) $\underline{x}^0$ and $\underline{h}$ are functions from Q into K.
4) E is a function from $Q \times Q \times U$ into K.
There we will show that $K-U$-automaton $A = (Q, E, \underline{x}^0, \underline{h})$ become a n-dimensional Linear Representation System $\sigma = ((K^n, F), x^0, h)$.
Let $x^0 := [x_1, x_2, \cdots, x_n]^T$ for the function $\underline{x}^0 : Q \to K; q_i \mapsto x_i (1 \leq i \leq n)$. Then x^0 may be in K^n. Let $h := [h_1, h_2, \cdots, h_n]$ for the function $\underline{h} : Q \to K; q_i \mapsto h_i (1 \leq i \leq n)$, then h may be in $K^{1\times n}$.
For the function $E : Q \times Q \times U \to K$, let a map $F : U \to K^{n\times n}$ saisfy the following equations:
$F_{i,j}(u) := E(q_i, q_j, u)$ for $1 \leq i \leq n, 1 \leq j \leq n$. Where $u \in U, q_i, q_j \in Q$ and $F_{i,j}(u)$ denote the element of i-th row and j-th column of matrix $F(u)$. Then F may be the same as E. Therefore, the $K-U$-automaton $A = (Q, E, \underline{x}^0, \underline{h})$ may be the same as the Linear Representation System $\sigma = ((K^n, F), x^0, h)$.
Eilenberg had defined the behavior $|A|(\omega) = \underline{h}E^*(\omega)\underline{x}^0$ for any $\omega \in \Omega$ and $E^* = I + E + E^2 + \cdots + E^n + \cdots$. There, we have $|A|(\omega) = a_\sigma(\omega)$ for any $\omega \in \Omega$. Namely, the behavior of A is the same as one of σ. By the above discussion, a $K-U$-automaton can be considered as a finite dimensional Linear Representation System.

(4.11) Theorem
A Linear Representation System $\sigma = ((K^n, F), x^0, h)$ is canonical if and only if the following conditions 1) and 2) hold.
1) rank $[x^0, F(u_1)x^0, \cdots, F(u_m)x^0, \cdots, F(u_1)^2x^0, F(u_1)F(u_2)x^0, \cdots, F(u_1)F(u_m)x^0, \cdots, F(u_m)^2x^0, \cdots, F(u_1)^{n-1}x^0, F(u_2)F(u_1)^{n-2}x^0, \cdots, F(u_m)^{n-1}x^0] = n$.
2) rank $[h^T, (hF(u_1))^T, \cdots, (hF(u_m))^T, (hF(u_1)^2)^T, \cdots, (hF(u_1)F(u_m))^T, \cdots, (hF(u_1)^{n-1})^T, (hF(u_1)^{n-2}F(u_m))^T, \cdots, (hF(u_m)^{n-1})^T] = n$.

[proof] See Propositions (4-B.6) and (4-B.13) in Appendix 4.5.

(4.12) Definition
Let the input value's set U be $U := \{u_i; 1 \leq i \leq m\}$ and let a map $\| \ \| : U \to N$ be $u_i \mapsto \|u_i\| = i$. And let a numerical value $|||\omega|||$ of an input $\omega \in \Omega$ be $|||\omega||| = \|\omega(|\omega|)\| + \|\omega(|\omega|-1)\| \times m + \cdots + \|\omega(1)\| \times m^{|\omega|-1}$ and $|||\mathit{1}||| = 0$.

And we define totally ordered relation by this numerical value in Ω. Namely, $\omega_1 \leq \omega_2 \iff |||\omega_1||| \leq |||\omega_2|||$.

(4.13) Definition
A canonical Linear Representation System $\sigma = ((K^n, F_s), \mathbf{e_1}, h_s)$ is said to be a quasi-reachable standard system if input sequences $\{\omega_i; 1 \leq i \leq n\}$ given by $\mathbf{e_i} = \phi_{F_s}(\omega_i)\mathbf{e_1}$ satisfy the following conditions:

1) $\mathit{1} = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i - 1$ for $i (1 \leq i \leq n)$ hold.

2) $\phi_{F_s}(\omega)\mathbf{e_1} = \sum_{i=1}^{j} \mathbf{e_j}$ holds for any input sequence such that $\omega_j < \omega < \omega_{j+1} (1 \leq i \leq n-1)$, $\omega \in \Omega$.

(4.14) Representation Theorem for equivalence classes
For any finite dimensional canonical Linear Representation System, there exists a uniquely determined isomorphic quasi-reachable standard system.

[proof] See (4-B.16) in Appendix 4.5.B.

(4.15) Definition
Let Y be a field K for convenience. A canonical Linear Representation System $\sigma_d = ((K^n, F_d), x_d^0, h_d)$ is said to be a distinguishable standard system if input sequences $\{\omega_i; 1 \leq i \leq n\}$ given by $\mathbf{e_1}^T = h_d \mathbf{e_1}^T \phi_{F_d}(\omega_i)$ satisfy the following conditions:
1) $\mathit{1} = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i - 1$ for $i (1 \leq i \leq n)$ hold.
2) $\mathbf{e_1}^T \phi_{F_d}(\omega) = \sum_{i=1}^{j} \alpha_i \mathbf{e_1}^T$ holds for any input sequence ω such that $\omega_j < \omega < \omega_{j+1} (1 \leq i \leq n-1)$.

(4.16) Representation Theorem for equivalence classes
For any finite dimensional canonical Linear Representation System, there exists a uniquely determined isomorphic distinguishable standard system.

[Proof] See (4-B.17) in Appendix 4.5.B.

(4.17) Definition
For any input response map $a \in F(\Omega, Y)$, the corresponding linear input/output map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ satisfies $A(\mathbf{e}_\omega)(\bar{\omega}) = a(\bar{\omega}|\omega)$ for $\omega, \bar{\omega} \in \Omega$.
Hence, A can be represented by the next infinite matrix H_a^L. This H_a^L is said to be a Hankel matrix of a.

$$H_a^L = \begin{array}{c} \\ \\ \\ \bar{\omega} \end{array} \left(\begin{array}{ccc} & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(\bar{\omega}\,|) \\ & & \end{array} \right)$$

See Remark 2 of Proposition (4-A.27) about the corresponding linear input/output map A.

(4.18) Theorem for existence criterions

For an input response map $a \in F(\Omega, Y)$, the following conditions are equivalent:

1) The input response map $a \in F(\Omega, Y)$ has the behavior of n-dimensional canonical Linear Representation System.

2) There exist n linearly independent vectors and no more than n linearly independent vectors in a set $\{S_l(\omega)a; |\omega| \leq n-1 \text{ for } \omega \in \Omega\}$.

3) The rank of the Hankel matrix H_a^L of a is n.

[proof] See (4-B.18) in Appendix 4.5.B.

Remark: Fliess [1974] has introduced the Hankel matrix of the non-commutative formal power series and shown that the recognizability of the formal power series is equal to the finite rank of its Hankel matrix.

Let $A(\Omega)$ have the following operation $\times$.

$\times : A(\Omega) \times A(\Omega) \to A(\Omega); (\sum_\omega \lambda(\omega)\mathbf{e}_\omega, \sum_{\bar{\omega}} \lambda(\bar{\omega})\mathbf{e}_{\bar{\omega}}) \mapsto (\sum_\omega \lambda(\omega)\mathbf{e}_\omega) \times (\sum_{\bar{\omega}} \lambda(\bar{\omega})\mathbf{e}_{\bar{\omega}})$
$= \sum_{\omega'} (\sum_{\omega|\bar{\omega}=\omega'} \lambda(\omega)\lambda(\bar{\omega})\mathbf{e}_{\omega'})$

Then $A(\Omega)$ is an algebra over K, and is a free algebra over K.

As $a \in F(\Omega, K)$ can be expressed as a formal power series $a = \sum_\omega a(\omega)\mathbf{e}_\omega$, it can be considered that $F(\Omega, K)$ contains $A(\Omega)$. Here, we introduce an operation $* : F(\Omega, K) \times F(\Omega, K); a \mapsto a^* = \sum_{j=1}^{\infty} (a - a(1))^j$.

(4.19) Definition

A smallest subset which is a sub algebra of $F(\Omega, K)$ which contains $A(\Omega)$ and is closed under the operation $*$ is said to be a rational set $Rat(\Omega)$. And $a \in Rat(\Omega)$ is said to be to be rational.

(4.20) Theorem
An input response function $a \in F(\Omega, K)$ is the behavior of a finite dimensional Linear Representation System if and only if the formal power series a is rational.

[proof] See $K-U$-automata (4.10) and Eilenberg [1974].

Remark 1: The equivalent condition for the commutative formal power series of one-variable to be rational has been obtained. See Gantmacher [1959] and Kalman, Falb and Arbib [1969]. Also see Matsuo and Hasegawa [1982] for two variables.
Remark 2: Fliess [1970],[1974] has pointed out that the recognizability of the three-variable commutative formal power series can be characterized by the same form as the rational function in Theorem (4.7).

(4.21) Theorem for a realization procedure
Let an input response $a \in F(\Omega, Y)$ satisfy the condition of Theorem (4.18), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), \mathbf{e_1}, h_s)$ which realizes the input response map a can be obtained by the following procedure:
1) Select the linearly independent vectors $\{S_l(\omega_i)a; 1 \leq i \leq n\}$ of the set $\{S_l(\omega)a; |\omega| \leq n-1, \omega \in \Omega\}$ in order of the index value.
2) Let the state space be K^n, the initial state be $\mathbf{e_1}$.
3) Let the output map $h_s = [a(\omega_1), a(\omega_2), a(\omega_3), \cdots, a(\omega_n)]$.
4) Let ${}_if_j \in K^n$ in $F_s(u_i) := [{}_if_1 \; {}_if_2 \; \cdots \; {}_if_n \;]$ be ${}_if_j := [{}_if_{j,1} \; {}_if_{j,2} \; \cdots \; {}_if_{j,n}]^T$, where $S_l(u_i)S_l(\omega_j)a = \sum_{k=1}^{j} {}_if_{1,k} \; S_l(\omega_k)a$, ${}_if_{j,k} \in K$ and
$\mathbf{e_1} = [100 \cdots 00]^T \in K^n$.

[proof] See (4-B.19) in Appendix 4.5.

(4.22) Example
Now we consider again the black-box discussed in Example (3.12). Here we treat it as Linear Representation Systems. We recall the black-box. Let $U := \{-1, 0\}$ and $Y := R$. The black-box may be given by the following equation:

$$\gamma(t+1) = \gamma^2(t) + \omega(t+1), \; t \in N, \; \gamma(0) = 0.$$

Where $\omega(t) \in U$ and $\gamma(t) \in Y$.

Let a be the input/output relation of it. Based on Theorem (4.21), we obtain a canonical Linear Representation System $\sigma = ((X, F_s), \mathbf{e_1}, h_s)$ which realizes a in the following manner:
1) Linearly independent vectors in a set $S_l(\Omega)a$ are obviously $a_1 := a$, $a_2 := S_l(-1)a$ and $a_3 := S_l(0|-1)a$.
2) Let a state set X be R^3 and an initial state x^0 be $\mathbf{e_1}$.
3) Let an output map $h_s : R^3 \to Y$ be $h_s = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$.
4) Since $S_l(-1)a_1 = a_2$, $S_l(-1)a_2 = a_1$ and $S_l(-1)a_3 = a_1$, $F_s(-1)$ can be given by the following. And since $S_l(0)a_1 = a_1$, $S_l(0)a_2 = a_3$ and $S_l(0)a_3 = a_3$, $F_s(0)$ is given by the following.

$$F_s(-1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_s(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(4.23) Example
Now we consider again the example discussed in Example (3.13). Here we treat it as Linear Representation Systems. Let's consider a hysteresis characteristic (Figure 3.4 in Example (3.13) of Chapter 3), which is given by the following relation:

$$\gamma(t) := \begin{cases} 0 & u < -\frac{\delta u}{2} \\ 1 & \frac{\delta u}{2} < u \\ \gamma(t-1) & -\frac{\delta u}{2} \leq u \leq \frac{\delta u}{2} \end{cases}$$

Where $u \in U, \gamma(t) \in Y = R$.

Let a be the input response map of the hysteresis characteristic. Applying Theorem (4.21), we can obtain the canonical Linear Representation System $\sigma = ((X, F_s), x^0, h_s)$ which realizes a in the following manner.
Let $U := \{u_1, u_2, u_3\}$. Where $-\frac{\Delta u}{2} \leq u_2 \leq \frac{\Delta u}{2}$, $\frac{\Delta u}{2} < u_1$ and $u_3 < -\frac{\Delta u}{2}$.
1) Linearly independent vectors in a set $S_l(\Omega)a$ are obviously $a_1 := a$ and $a_2 := S_l(u_1)a$.
2) Let a state set X be R^2 and an initial state x^0 be $\mathbf{e_1}$.
3) Let an output map $h_s : R^2 \to Y$ be $h_s = \begin{bmatrix} 0 & b \end{bmatrix}$.
4) Since $S_l(u_1)a_1 = a_2$ and $S_l(u_1)a_2 = a_2$, $F_s(u_1)$ can be given by the following. Since $S_l(u_2)a_1 = a_1$ and $S_l(u_2)a_2 = a_2$, $F_s(u_2)$ can be given by

the following. And since $S_l(u_3)a_1 = a_1$ and $S_l(u_3)a_2 = a_1$, $F_s(u_3)$ can be given by the following.

$$F_s(u_1) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ F_s(u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\ F_s(u_3) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

4.3 Partial Realization Theory of Linear Representation Systems

Here we consider a partial realization problem by multi-experiment. Let $\underline{a}$ be an $\underline{N}$ sized input response map($\in F(\Omega_{\underline{N}}, Y)$), where $\underline{N} \in N$ and $\Omega_{\underline{N}} := \{\omega \in \Omega; |\omega| \leq \underline{N}\}$. The $\underline{a}$ is said to be a partial input response map. A finite dimensional Linear Representation System $\sigma = ((X, F), x^0, h)$ is called a partial realization of $\underline{a}$ if $h\phi_F(\omega)x^0 = \underline{a}(\omega)$ holds for any $\omega \in \Omega_{\underline{N}}$.

A partial realization problem of Linear Representation Systems can be stated as follows:
< For any given $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, find a partial realization σ of $\underline{a}$ such that the dimensions of state space X of σ is minimum, where the σ is said to be a minimal partial realization of $\underline{a}$. Moreover, show when the minimal realizations are isomorphic.>

(4.24) Proposition
For any given $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, there always exists a minimal partial realization of it.

[proof] For any $\omega \notin \Omega_{\underline{N}}$, set $\underline{a}(\omega) = 0$. Then $\underline{a} \in F(\Omega, Y)$, and Theorem (4.18) implies that there exists a finite dimensional partial realization of $\underline{a}$. Therefore, there exists a minimal partial realization.

Minimal partial realizations are, in general, not unique modulo isomorphism. Therefore, we introduce a natural partial realization, and we show that natural partial realizations exist if and only if they are isomorphic.

(4.25) Definition
For a Linear Representation System $\sigma = ((X, F), x^0, h)$ and some $p \in N$, if $X = \ll \{\phi_F(\omega)x^0; \omega \in \Omega_p\} \gg$, then σ is said to be p-quasi-reachable.
Where $\ll S \gg$ denotes the smallest linear space which contains a set S.
Let q be some integer. If $h\phi_F(\omega)x = 0$ implies x=0 for any $\omega \in \Omega_q$, then σ is said to be q-distinguishable.

For a given $\underline{a} \in F(\Omega_L, Y)$, if there exist p and $q \in N$ such that $p+q < L$ and σ is p-quasi-reachable and q-distinguishable then σ is said to be a natural partial realization of $\underline{a}$.
For a partial input response map $\underline{a} \in F(\Omega_L, Y)$, the following matrix $H^L_{\underline{a}\,(p,L-p)}$ is said to be a finite-sized Hankel-matrix of $\underline{a}$.

$$H^L_{\underline{a}\,(p,L-p)} = \begin{pmatrix} & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(\bar{\omega}\,|) \end{pmatrix}$$
$$\bar{\omega}$$

Where $\bar{\omega} \in \Omega_p$ and $\omega \in \Omega_{L-p}$.

(4.26) Theorem
Let $H^L_{\underline{a}\,(p,L-p)}$ be the finite Hankel-matrix of $\underline{a} \in F(\Omega_L, Y)$. Then there exists a natural partial realization of $\underline{a}$ if and only if the following conditions hold:
rank $H^L_{\underline{a}\,(p,L-p)}$= rank $H^L_{\underline{a}\,(p,L-p-1)}$ =rank $H^L_{\underline{a}\,(p+1,L-p-1)}$ for some $p \in N$.

[proof] See (4-C.9) in Appendix 4.5.C.

(4.27) Theorem
There exists a natural partial realization of a given partial input response map $\underline{a} \in F(\Omega_L, Y)$ if and only if the minimal partial realization of $\underline{a}$ are unique modulo isomorphism.

[proof] See (4-C.11) in Appendix 4.5.C.

(4.28) Theorem
Let a partial input response $\underline{a} \in F(\Omega_L, Y)$ satisfy the condition of Theorem (4.26), then the quasi-reachable standard system $\sigma_s = ((X, F_s), \mathbf{e_1}, h_s)$ which realizes $\underline{a}$ can be obtained by the following algorithm.
Set $n :=$ rank $H^L_{\underline{a}\,(p,L-p)}$, where $H^L_{\underline{a}\,(p,L-p)}$ is the finite Hankel-matrix of $\underline{a} \in F(\Omega_L, Y)$.

1) Select the linearly independent vectors $\{S_l(\omega_i)\underline{a} \in F(\Omega_{L-p}, Y); 1 \le i \le n\}$ from $H^L_{\underline{a}\,(p,L-p)}$ in order of the numerical value.
2) Let the state space be K^n, the initial state be $\mathbf{e_1} = [100, \cdots, 0]^T$.
3) Let the output map $h_s = [\underline{a}(\mathit{1})\underline{a}(\omega_2)\underline{a}(\omega_3)\cdots\underline{a}(\omega_n)]$.
4) Let ${}_if_j$ in $F_s(u_i) := [{}_if_1\ {}_if_2\ \cdots\ {}_if_n\ ]$ be ${}_if_j := [{}_if_{j,1}\ \mathrm{i}_if_{j,2}\ \cdots\ {}_if_{j,n}]$ for $1 \le i \le n$. Where ${}_if_j$ is given by the following.
$\underline{S_l}(u_i)\underline{S_l}(\omega_j)\underline{a} = \sum_{k=1}^{j} {}_if_{j,k}\underline{S_l}(\omega_i)\underline{a}$, ${}_if_{j,k} \in K$ in the sense of $F(\Omega_{L-p}, Y)$ and $\underline{S_l}(\omega) : F(\Omega_s, Y) \to F(\Omega_{s-|\omega|}, Y)$; $\underline{a} \mapsto \underline{S_l}(\omega)\underline{a}[; \bar{\omega} \mapsto \underline{a}(\bar{\omega}|\omega)]$.

[proof] See (4-C.12) in Appendix 4.5.C.

4.4 Historical Notes and Concluding Remarks

There may be a sign of notions of Linear Representation Systems in Sussman [1976]. As we mentioned in (4.9) and (4.10), homogeneous bilinear systems and $K-U$ automaton are a sort of Linear Representation Systems. See Tarn & Nonoyama [1976] and Fliess [1978] for homogeneous bilinear systems of discrete-time. See also Brockett [1976a] for ones of continuous-time. See Paz [1966] for probablistic automaton. Schuzenberger [1961] considered generalized automata which are called $K - U$ automata by Eilenberg [1974].
$\mathrm{A}(\Omega)$ in Example (4.2) may be equal to the algebra of polynomial of non commutative variable introduced by Fliess [1970] and [1974].

Nerode equivalence for $A(\Omega)$ in Theorem (4.3) is new. Therefore, nerode equivalence for $A(\Omega)$ in Theorem (4.4) is new. See Section 3.4 in Chapter 3 for the comments for nerode equivalence.

It is shown that the uniqueness Theorem (4.5) holds in the sense of Linear Representation Systems, namely the theorem is more stronger than in the sense of General Dynamical Systems. We also can give the same Theorem (5.14) as Theorem (4.5) by using different dynamical systems which is said to be Affine Dynamical Systems. Therefore, we obtain two theorems for arbitrary input response map (equivalently, input/output map with causality). Details of relations between Linear Representation Systems and Affine Dynamical Systems were discussed in Niinomi and Matsuo [1981].

Theorem (4.11) is the theorem for finite dimensional Linear Representation Systems to be canonical. It can be easily understood that this theorem is an extension of theorem for finite dimensional linear systems to be canonical.

We gave the quasi-reachable standard system and distinguishable standard system that correspond to companion forms of linear systems.

We gave the two criterions for the behavior of finite dimensional Linear Representation Systems. See the Remark in Theorems (4.18) and (4.20) regarding this.

We also gave a realization procedure to obtain the quasi-reachable standard system from a given input response map.

For other nonlinear systems, Sontag [1979a] has given a procedure to obtain from a given input/output map. but the procedure is not as clear as ours. Moreover, only we obtained partial Realization Theorems (4.26) and (4.27). Also we obtained a partial realization algorithm.

In discrete-time case, Isidori [1975] has only given a sufficient condition of uniqueness and an algorithm for inhomogeneous bilinear systems.

4.5 Appendix

This Appendix is prepared for the proof of any results about Linear Representation Systems.

4.5.A Realization Theorem

This Appendix 4.5.A is prepared for the proof of Realization Theorem (4.5) of Linear Representation Systems. To prove it, we equivalently convert the Linear Representation Systems to sophisticated Linear Representation Systems owing to the result of Appendix 4.5.A to 4.5.C. In Appendix 4.5.A, we prove the realization theorem in the sophisticated Linear Representation Systems. This implies that Theorem (4.5) is proved.

4.5.A.1 Linear State STRUCTURE: Linear U-Actions

(4-A.1) Definition
A system given by the following equation is written as a pair (X, F) and it is said to be a linear U-action.
$x(t+1) = F(\omega(t+1)x(t)$
Where X is a linear space over the field K and a map $F : U \to L(X); u \mapsto F(u)$.
Let (X_1, F_1) and (X_2, F_2) be linear U-actions, then a linear map $T : X_1 \to X_2$ is said to be a U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ if T satisfies $TF_1(u) = F_2(u)T$ for any $u \in U$.

(4-A.2) Example
Let $A(\Omega)$ and S_r be the same as that considered in Example (4.2). Then $(A(\Omega), S_r)$ is a linear U-action.

(4-A.3) Example
In the set $F(\Omega, Y)$ of any input response map, let S_l be the same as in Example (4.2). Then $(F(\Omega, Y), S_l)$ is a linear U-action.

(4-A.4) Example
Let $X = K^n$, and $F(u)$ belong to $K^{n\times n}$ for any $u \in U$, (K^n, F) is a linear U-action, and it represents a state difference equation:
$x(t+1) = F(\omega(t+1)x(t)$.

(4-A.5) Definition
For linear U-actions $(A(\Omega), S_r)$ and $(F(\Omega, Y), S_l)$ considered in Examples (4-A.2) and (4-A.3), a linear U-morphism $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ is said to be a linear input/output map. For a linear U-action (X, F), a U-morphism $G : (A(\Omega), S_r) \to (X, F)$ is said to be a linear input map, and a U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$ is said to be a linear observation map.

Remark: A linear input/output map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ is different from the map discussed in Perlman [1980].

As the structure of General Dynamical Systems, we have introduced U-actions and the Ω-modules in the Appendix 3.5 of Chapter 3. And we have clarified the relation between U-actions and Ω-modules. Here we newly introduce linear Ω-modules and $A(\Omega)$-modules. Then we relate the connection among linear U-actions, linear Ω-modules and $A(\Omega)$-modules.

(4-A.6) Definition
Let X be a linear space over the field K and a map $\phi : \Omega \to L(X)$ be a monoid morphism. i.e. $\phi(1) = I$ (the Identity map on X), and $\phi(\omega_2|\omega_1) = \phi(\omega_2) \cdot \phi(\omega_1)$ hold for any $\omega_1, \omega_2 \in \Omega$. Then a pair (X, ϕ) is said to be a linear Ω-module.

Remark: Let (X, ϕ) be a linear Ω-module. For $\omega \in \Omega$ and n:=$|\omega|$, a linear map $\phi(\omega) : X \to X$ transfer a state $x(t)$ at time t to a state $x(t+n)$ after n time and it is linear transformation. Therefore, it represents a linear state transition equation.
$x(t+n) = \phi(\omega)x(t)$

(4-A.7) Proposition
Let X be a linear space over the field K. For any map $F : U \to L(X)$, a map $\phi : \Omega \to L(X)$ obtained by the following formula *) is a monoid morphism. Moreover, this correspondence is bijective.
$*)$:

$$\begin{cases} \phi(\omega) = F(\omega(|\omega|) \cdot F(\omega(|\omega| - 1) \cdot F(\omega(1) \\ \phi(1) = I \end{cases}$$

for any $\omega \in \Omega$.

Therefore, a linear U-action (X, F) injectively corresponds to a linear Ω-module (X, ϕ) by the formula *). Let (X, F) be a linear U-action corresponding to a linear Ω-module (X, ϕ), then (X, F) may be written as (X, ϕ_F). Conversely, (X, ϕ_F) may be written as (X, F).

(4-A.8) Example
For the linear U-action $(A(\Omega), S_r)$ considered in Example (4-A.2), linear Ω-module $(A(\Omega), S_r)$ corresponding to it is given by setting $S_r(\omega) := e_\omega$ for $\omega \in \Omega$.

(4-A.9) Example
For the linear U-action $(F(\Omega, Y), S_l)$ considered in Example (4-A.3), the linear Ω-module corresponding to it is given by $(F(\Omega, Y), S_l)$, where $S_l(\omega)$: $F(\Omega, Y) \to F(\Omega, Y); a \mapsto S_l(\bar{\omega})a[; \mapsto a(\bar{\omega}|\omega)]$.
Here we introduce $A(\Omega)$-module. We discuss a connection among linear U-actions, linear Ω-modules and $A(\Omega)$-modules.

(4-A.10) Definition
Let X be a linear space over K and $\tilde{\phi} : A(\Omega) \to L(X)$ be an algebra morphism. i.e. $\tilde{\phi}(x \cdot y) = \tilde{\phi}(x) \cdot \tilde{\phi}(y)$ and $\tilde{\phi}(\mathbf{e_1}) = I$ hold. Then a pair $(X, \tilde{\phi})$ is said to be an $A(\Omega)$-module.

Let's introduce a map $:A(\Omega) \times A(\Omega) \to A(\Omega); (\sum_\omega \lambda(\omega)\mathbf{e}_\omega, \sum_{\bar{\omega}} \lambda(\bar{\omega})\mathbf{e}_{\bar{\omega}}) \mapsto$ $(\sum_\omega \lambda(\omega)\mathbf{e}_\omega) \times$ $(\sum_{\bar{\omega}} \lambda(\bar{\omega})\mathbf{e}_{\bar{\omega}}) = \sum_{\omega'} (\sum_{\omega|\bar{\omega}=\omega'} \lambda(\omega)\lambda(\bar{\omega})\mathbf{e}_{\omega'})$
Then $A(\Omega)$ is an algebra and the free algebra over Ω. Next we state an important lemma about the free algebra.

(4-A.11) Lemma
Let A be any algebra and a map $e : \Omega \to A(\Omega)$ be $; \omega \mapsto e_\omega$. For any monoid morphism $f : \Omega \to A$, there uniquely exists $\tilde{f} : A(\Omega) \to A$ such that $f = \tilde{f}(\lambda) \cdot e$. Moreover, $\tilde{f}(\lambda) = (\sum_\omega \lambda(\omega)\phi(\omega))$ holds for any $\lambda =$ $(\sum_\omega \lambda(\omega)e_\omega) \in A(\Omega)$. Conversely, for any algebra morphism $\tilde{f} : A(\Omega) \to A$, $f = \tilde{f} \cdot e$ is a monoid morphism $: \Omega \to A$.

(4-A.12) Proposition
Let X be a linear space, $Mor(\Omega, L(X))$ be a set of any monoid morphisms $: \Omega \to L(X)$ and let $AMor(A(\Omega), L(X))$ be a set of any algebra morphisms $: A(\Omega) \to L(X)$.

Then a map $\tilde{:}Mor(\Omega, L(X)) \to AMor(A(\Omega), L(X)); \phi \mapsto \tilde{\phi}\ [; \lambda \mapsto (\sum_\omega \lambda(\omega)\phi(\omega))]$ is bijective.

[proof] $L(X)$ is a linear space and an algebra by composition's operation. Let A in Lemma (4-A.11) be $L(X)$, then we obtain this proposition.

(4-A.13) Lemma
The $A(\Omega)$-module corresponding to $(A(\Omega), S_r)$ is $(A(\Omega), \cdot)$.
Where $(A(\Omega), S_r)$ have been considered in Example (4-A.2).

[proof] $A(\Omega)$ is clearly an algebra. By Proposition (4-A.7), the linear morphism $S_r : \Omega \to L(A(\Omega))$ corresponding to $S_r : U \to L(A(\Omega)); u \mapsto e_u\cdot$ is given by $S_r(\omega) = e_\omega\cdot$ for any $\omega \in \Omega$. By Proposition (4-A.12), the algebra morphism $\tilde{S_r} : A(\Omega) \to L(A(\Omega))$ corresponding to $S_r : \Omega \to L(A(\Omega))$ is given by $\tilde{S_r}(\lambda) = \lambda\cdot$ for $\lambda \in A(\Omega)$.

(4-A.14) Example
For the linear Ω-module $(F(\Omega, Y), S_l)$ considered in (4-A.9), the $A(\Omega)$-module $(F(\Omega, Y), \tilde{S_l})$ corresponding to it is given by setting $\tilde{S_l}(\mathbf{e}_\omega) := S_l(\omega)$.

(4-A.15) Definition
Let (X_1, F_1) [(X_1, ϕ_1) or $(X_1, \tilde{\phi}_1)$] and (X_2, F_2) [(X_2, ϕ_2) or $((X_2, \tilde{\phi}_2)$ be linear U-actions [linear Ω -module or $A(\Omega)$-module], then a linear map $T : X_1 \to X_2$ is said to be a linear U-morphism $(X_1, F_1) \to (X_2, F_2)$ [a linear U-morphism $(X_1, \phi_1) \to (X_2, \phi_2)$ or an $A(\Omega)$-morphism $(X_1, \tilde{\phi}_1) \to (X2, \tilde{\phi}_2)$ if T satisfies $TF_1(u) = F_2(u)T$ for any $u \in U$ [$T\phi_1(\omega) = \phi_2(\omega)T$ for any $\omega \in \Omega$ or $T\tilde{\phi}_1(e_\omega) = \tilde{\phi}_2(e_\omega)T$ for any $\omega \in \Omega$].

(4-A.16) Proposition
Let $(X, \tilde{\phi})$ be an $A(\Omega)$-module. A map T is an $A(\Omega)$-morphism $T : (A(\Omega), \cdot)$ $\to (X, \tilde{\phi})$ if and only if $T(\lambda) = \tilde{\phi}(\lambda)T(\mathbf{e_1})$ holds.

[proof] Let T be an $A(\Omega)$-morphism, i.e., T satisfy $T(\lambda) = \phi(\lambda)T$. Then $T(\lambda) = T(\lambda\cdot\mathbf{e_1}) = \tilde{\phi}(\lambda))T(\mathbf{e_1})$ holds. Conversely, let $T(\lambda) = \tilde{\phi}(\lambda)T(\mathbf{e_1})$ hold. Since T is an algebra morphism, $T(\lambda \cdot \tilde{\lambda}) = \tilde{\phi}(\lambda) \cdot \tilde{\phi}(\tilde{\lambda})T(\mathbf{e_1}) = \tilde{\phi}(\lambda)T(\tilde{\lambda})$ holds for any $\lambda, \tilde{\lambda} \in A(\Omega)$. i.e., $T(\lambda\cdot) = \tilde{\phi}(\lambda)T$ holds. And clearly T is a linear operator. Therefore, T is an $A(\Omega)$-morphism $T : (A(\Omega), \cdot) \to (X, \tilde{\phi})$.

(4-A.17) Proposition
Let $i = 1$ and $i = 2$. Let (X_i, F_i) be a linear U-action, (X_i, ϕ_i) be a linear Ω-module corresponding to it. And let $(X_i, \tilde{\phi}_i)$ be an $A(\Omega)$-module corresponding to (X_i, ϕ_i). Then the following three conditions are equivalent.
1) A morphism T is a linear U-morphism : $(X_1, F_1) \to (X_2, F_2)$.
2) A morphism T is a linear Ω-morphism : $(X_1, \phi_1) \to (X_2, \phi_2)$.
3) A morphism T is a $A(\Omega)$-morphism : $(X_1, \tilde{\phi}_1) \to (X_2, \tilde{\phi}_2)$.

[proof] By Proposition (4-A.12), it is obtained that condition 1) is equivalent to condition 2). Here, we assist that 2) is equivalent to 3).
Let's assume that condition 2) holds, i.e., $T\phi_1(\omega) = \phi_2(\omega)T$ holds for any $\omega \in \Omega$. For any $\lambda = \sum_\omega \lambda(\omega)e_\omega \in A(\Omega)$, Proposition (4-A.12) implies that $T\tilde{\phi}_1(\omega) = T\{\sum_\omega \phi_1(\omega)\} = \lambda(\omega)T\phi(\omega) = \sum_\omega \lambda(\omega)\{T\phi(\omega)\} = \tilde{\phi}_2(\lambda)T$ holds. Therefore, condition 3) holds. Conversely, let condition 3) hold, i.e., $T\tilde{\phi}_1(e_\omega) = \tilde{\phi}_2(e_\omega)T$ hold for any $\omega \in \Omega$. Then Proposition (4-A.12) implies that $T\phi_1(\omega) = \phi_2(\omega)T$ holds. Therefore T is a linear Ω -morphism : $(X_1, \phi_1) \to (X_2, \phi_2)$.

In Appendix of Chapter 3, we have introduced the sub U-action, the quotient U-action and the product U-action. Here we introduce them with linear structure, i.e., sub linear U-actions, quotient linear U-actions and product linear U-actions.

(4-A.18) Sub linear U-actions
Let (X, F) be a linear U-action and $Y \subseteq$ be invariant sub-space under F, i.e., $F(u)y \in Y$ for any $u \in U$ and any $y \in Y$. Let $F_Y(u) := F(u)|_Y$ (restriction of the map $F(u)$ to Y), then (Y, F_Y) is a linear U-action, and it is said to be a sub linear U-action of (X, F).

(4-A.19) Quotient linear U-action
Let (X, F) be a linear U-action and a linear equivalence relation R in X be consistent with F. Namely, an equivalence relation R is given by $x_1 \; R \; x_2 \iff x_1 - x_2 \in S$ for some linear sub space $S \subseteq X$, and $x_1 \; R \; x_2$ implies $F(u)x_1 \; R \; F(u)x_2$ for any $u \in U$. Then we can consider a quotient linear space $X/_R = X/S$. Therefore, we can obtain a quotient linear U-action $(X/Z, \tilde{F})$. Where $\tilde{F}(u) : X/Z \to X/Z; [x] \mapsto [F(u)x]$ for any $u \in U$.

(4-A.20) Corollary
Any linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ can be normally decomposed into $X_1 \xrightarrow{\pi} X_1/_{\ker\, T} \xrightarrow{T^b} \mathrm{im}\; T \xrightarrow{j} X_2$

Where $X_1/_{\ker T}$ is the quotient set induced by the map T, π is the canonical surjection, T^b is the bijection associated with T and j is the canonical injection. Moreover, they are also linear U-morphism respectively.

(4-A.21) Product linear U-actions
Let (X_1, F_1) and (X_2, F_2) be linear U-actions and define $(F_1 \times F_2)(u) : X_1 \times X_2 \to X_1 \times X_2; (x_1, x_2) \mapsto (F_1(u)x_1, F_2(u)x_2)$ for the product space $X_1 \times X_2$ and any $u \in U$. Therefore, $(X_1 \times X_2, F_1 \times F_2)$ is a linear U-action, it is said to be a product linear U-action of (X_1, F_1) and (X_2, F_2).

(4-A.22) Proposition
$A(\Omega)' = F(\Omega, Y)$.
Where $A(\Omega)'$ is a set of any linear map from $A(\Omega)$ to Y.

[proof] For any $a \in F(\Omega, Y)$, set $\tilde{} : a \mapsto \tilde{a}[; \sum \lambda(\omega)e_\omega \mapsto \sum \lambda(\omega)a(\omega)]$, then $\tilde{a} \in A(\Omega)'$ holds. For any $\tilde{a} \in A(\Omega)'$, set $e* : \tilde{a} \mapsto \tilde{a} \cdot e[; \omega \mapsto (e_\omega)$, then $\tilde{a} \cdot e \in F(\Omega, Y)$ holds. Here, $e * \cdot \tilde{=} I$ and $\tilde{\cdot} e* = I$ hold. Since $F(\Omega, Y)$ is a concrete expression of $A(\Omega)'$, we obtain $A(\Omega)' = F(\Omega, Y)$.

4.5.A.2 Pointed Linear U-Actions

In Appendix 3.5.B of Chapter 3, we have introduced the pointed U-action and U-actions with an input map, and we have shown that they are equivalent. In this section, we introduce pointed linear U-actions and linear U-actions with a linear input map, and show that they are equivalent. Moreover, we discuss reachability of linear U-actions.

(4-A.23) Definition
For a linear U-action (X, F) and an initial state $x^0 \in X$, a collection $((X, F), x^0)$ is said to be a pointed linear U-action. A pointed linear U-action $((X, F), x^0)$ represents the following equations:

$$\begin{cases} x(t+1) = F((t+1))x(t) \\ x(0) = x^0 \end{cases}$$

For any $t \in N$, where x(t) $\in X$ and $\omega(t) \in U$.

For the reachable set $R(x^0) = \{\phi_F(\omega)x^0; \omega \in \Omega\}$, the smallest linear space which contains $R(x^0)$ is equal to X, then $((X, F), x^0)$ is said to be quasi-reachable.

(4-A.24) Example
For the linear U-action $(A(\Omega), S_r)$ considered in Example (4-A.2) and the unit element $\mathbf{e_1}$ of multiplication, $((A(\Omega), S_r), \mathbf{e_1})$ is a pointed linear U-action and quasi-reachable.

(4-A.25) Example
For the linear U-action $(F(\Omega, Y), S_l)$ considered in Example (4-A.3) and an input response map $a \in F(\Omega, Y)$, $((F(\Omega, Y), S_l), a)$ is a pointed linear U-action.

(4-A.26) Definition
For pointed linear U-actions $((X_1, F_1), x_1^0)$ and $((X_2, F_2), x_2^0)$, a linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $Tx_1^0 = x_2^0$ is said to be a pointed linear U-morphism : $((X_1, F_1), x_1^0) \to ((X_2, F_2), x_2^0)$.

(4-A.27) Proposition
For any pointed linear U-action $((X, F), x^0)$, there exists a unique pointed linear U-morphism $G : ((A(\Omega), S_r), \mathbf{e_1}) \to ((X, F), x^0)$.
The G is said to be a linear input map.

[proof] According to setting $G(\mathbf{e_1}) = x^0$ in Proposition (4-A.16) and $\{e_\omega; \omega \in \Omega\}$ being the basis in $A(\Omega)$, G is unique.

Remark 1: According to Propositions (4-A.17) and (4-A.27), a linear input map $G : (A(\Omega), S_r) \to (X, F)$ corresponds to an initial state $x^0 \in X$ uniquely and this correspondence is isomorphism.
Remark 2: If a pointed linear U-action $((X, F), x^0)$ in Proposition (4-A.27) is replaced with $((F(\Omega, Y), S_l), a)$ considered in Example (4-A.3), then a linear input map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ corresponds to an input response map $a \in F(\Omega, Y)$ uniquely, and this correspondence is isomorphism.

By definition of quasi-reachability and the formula of linear input map, the following proposition can be obtained easily:

(4-A.28) Proposition
A pointed linear U-action $((X, F), x^0)$ is quasi-reachable if and only if the corresponding linear input map G is surjective.

4.5.A.3 Linear U-Actions with a Readout Map

In Appendix 3.5 of Chapter 3, we have introduced the U-action with a readout map and U-actions with an output map, and we have shown that they are equivalent. In this section, we introduce linear U-actions with a readout map and linear U-actions with a linear output map, and show that they are equivalent. Moreover, we discuss distinguishablity of linear U-actions with a readout map.

(4-A.29) Definition
For a linear U-action (X, F) and a linear map $h : X \to Y$, a collection $((X, F), h)$ is said to be a linear U-action with a readout map. A linear U-action with a readout map $((X, F), h)$ represents the following equations:

$$\begin{cases} x(t+1) = F((t+1))x(t) \\ \gamma(t) = hx(t) \end{cases}$$

For any $t \in N$, where $x(t) \in X$ and $\gamma(t) \in Y$.

For any $\omega \in \Omega$, if $h\phi_F(\omega)x_1 = h\phi_F(\omega)x_2$ implies $x_1 = x_2$, then $((X, F), h)$ is said to be distinguishable.
Let $((X_1, F_1), h_1)$ and $((X_2, F_2), h_2)$ be linear U-actions with a readout map, then a linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $h_1 = h_2T$ is said to be a linear U-morphism with a readout map $T : ((X_1, F_1), h_1) \to ((X_2, F_2), h_2)$.

(4-A.30) Example
For the linear U-action $(A(\Omega), S_r)$ considered in (4-A.2) and any input response map $a \in F(\Omega, Y)$, $((A(\Omega), S_r), a)$ is a linear U-action with a readout map (see Proposition (4-A.22)).

(4-A.31) Example
Regarding the linear U-action $(F(\Omega, Y), S_l)$ in Example (4-A.3), by defining a linear map $1 : (F(\Omega, Y) \to Y; a \mapsto a(1)$, $((F(\Omega, Y), S_l), 1)$ is a linear U-action with a readout map and it is distinguishable.

(4-A.32) Proposition
For any linear U-action with a readout map $((X, F), h)$, there exists a unique linear U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$ which satisfies $h = 1EH$,

where $(Hx)(\omega) = h\phi_F(\omega)x$ holds for any $x \in X$, $\omega \in \Omega$.
The H is said to be a linear observation map.

[proof] Let $((X, F), h)$ be any linear U-action with a readout map. Defining $(Hx)(\omega) = h\phi_F(\omega)x$ (for any $x \in X$, $\omega \in \Omega$), we can obtain a linear observation map $H : (X, F) \to (F(\Omega, Y), S_l)$ and H satisfies $h = 1EH$. Next, we will show uniqueness. Let H be a linear observation map : $(X, F) \to (F(\Omega, Y), S_l)$ which satisfies $h = 1EH$, then $(Hx)(\omega) = S_l(\omega)Hx(1) = 1S_l(\omega)Hx) = 0(H\phi_F(\omega)x) = h\phi_F(\omega)x$ holds for any $x \in X$, $\omega \in \Omega$. Therefore, H is unique.

Remark 1: According to Proposition (4-A.32), a linear observation map $H : (X, F) \to (F(\Omega, Y), S_l)$ corresponds uniquely to a linear map $h : X \to Y$ and this correspondence is isomorphism.
Remark 2: If $((X, F), h)$ in Proposition (4-A.32) is replaced with $((A(\Omega)), S_r), a)$ considered in Example (4-A.30), a linear observation map: $(A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ is a linear input/output map.

A following proposition is obtained easily by noticing the definition of distinguishability and Proposition (4-A.32).

(4-A.33) Proposition
A linear U-action with a readout map $((X, F), h)$ is distinguishable if and only if the corresponding linear observation map $H : (X, F) \to (F(\Omega, Y), S_l)$ is injective.

4.5.A.4 Linear Representation Systems

In this section, we introduce sophisticated Linear Representation Systems, and show that Linear Representation Systems (said to be a naive Linear Representation Systems) introduced in section 4.1 and sophisticated Linear Representation Systems are considered as the same thing.

(4-A.34) Definition
A collection $\Sigma = ((X, F), G, H)$ is said to be a sophisticated Linear Representation System, if G is a linear input map : $((A(\Omega), S_r) \to (X, F)$ and H is a linear observation map : $(X, F) \to (F(\Omega, Y), S_l)$.
A linear input/output map $A_\Sigma := H \cdot G : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$ is said to be the behavior of Σ.

For a linear input/output map A, if $A_\Sigma = A$, then sophisticated Linear Representation System Σ is called a realization of A.
A sophisticated Linear Representation System $\Sigma = ((X, F), G, H)$ is called canonical if G is surjective and H is injective.
For $\Sigma_1 = ((X_1, F_1), G_1, H_1)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2)$, a linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $TG_1 = G_2$ and $H_1 = H_2T$ is said to be a sophisticated Linear Representation System morphism : $\Sigma_1 \to \Sigma_2$. If T is surjective and injective then $T : (X_1, F_1) \to (X_2, F_2)$ is said to be an isomorphism.

(4-A.35) Example
For the linear U-action $(A(\Omega), S_r)$ in Example (4-A.2), identity map I on $A(\Omega)$ and a linear input/output map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$, a collection $((A(\Omega), S_r), I, A)$ is a sophisticated Linear Representation System with the behavior A.
For the linear U-action $(F(\Omega, Y), S_l)$ in Example (4-A.3), a linear input/output map A and identity map I on $F(\Omega, Y)$, then a collection $((F(\Omega, Y), S_l), A, I)$ is a sophisticated Linear Representation System with the behavior A.
In this situation, we consider the relation between sophisticated Linear Representation Systems and naive ones.

(4-A.36) Proposition
For any sophisticated Linear Representation System $\Sigma = ((X, F), G, H)$, there exists a unique naive Linear Representation System $\sigma = ((X, F), x^0, h)$ corresponding to the sophisticated Linear Representation System by two equations (a.1) and (a.2).
$\sum_\omega \lambda(\omega)\phi_F(\omega)x^0 = G(\sum_\omega \lambda(\omega)S_l(\omega))$ .... (a.1)
$h\phi_F(\omega)x = (Hx)(\omega)$ for any $x \in X$, $\omega \in \Omega$ .... (a.2)
This correspondence is isomorphic in the category's sense (Pareigis [1970]).

[proof] It is easily obtained from Remark 1 of Proposition (4-A.25) and Remark 1 of Proposition (4-A.32).

4.5.A.5 (Sophisticated) Realization Theorem

In this section, we will prove the Realization Theorem (4.5). According to Remark 2 in Proposition (4-A.27) (or Remark 2 in Proposition (4-A.32)) and Proposition (4-A.36), the realization theorem can be replaced with the following Theorem (4-A.37). Therefore, proving this theorem implies proving Realization Theorem (4.5).

(4-A.37) (Sophisticated) Realization Theorem
For any linear input/output map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$, there exist at least two sophisticated canonical Linear Representation Systems that realize A (existence part).

Let $\Sigma_1 = ((X_1, F_1), G_1, H_1)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2)$ be sophisticated canonical Linear Representation Systems that have the same behavior, then there exists an isomorphism $T : \Sigma_1 \to \Sigma_2$ (uniqueness part).

[proof] A next Corollary (4-A.38) signifies proving the existence part. Moreover, Remark in Corollary (4-A.42) signifies proving the uniqueness.

(4-A.38) Corollary
For any linear input/output map $A : (A(\Omega), S_r) \to (F(\Omega, Y), S_l)$, the following sophisticated Linear Representation Systems (1) and (2) are canonical realizations of A.
(1) $\Sigma_q = ((A(\Omega)/\ker A, \tilde{S}_l, \pi, A^i)$.
Where π is the canonical surjection : $A(\Omega) \to A(\Omega)/_{\ker A}$ and A^i is given by $A^i = jA^b$ for $A^b : A(\Omega)/_{\ker A} \to \text{im } A$ being isomorphic with A and j being the canonical injection : $\text{im } A \to F(\Omega, Y)$.
(2) $\Sigma_s = ((\text{im } A, S_l), A^s, j)$.
Where $A^s = A^b \cdot j$.

[proof] This can be obtained easily by Corollary (4-A.20), Example (4-A.35), the definition of canonicality and the definition of the behavior.

Next, to prove the uniqueness part of Theorem (4-A.37), we introduce a following morphism $Mor(\Sigma_1, \Sigma_2)$ from a sophisticated Linear Representation System Σ_1 to another sophisticated Linear Representation System Σ_2. Where Σ_1 and Σ_2 are given by $\Sigma_1 = ((X_1, F_1), G_1, H_1)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2)$ respectively.
$Mor(\Sigma_1, \Sigma_2) := \{$ a relation $T : X_1 \to X_2; GrT_{12}^{min} \subseteq GrT_{12} \subseteq GrT_{12}^{max}\}$.
Where GrT_{12}^{min}, GrT_{12} and GrT_{12}^{max} denote the graph of $T_{12}^{min} := G_2 \cdot G_1^{-1}$, T_{12} and $H_2^{-1} \cdot H_1$ respectively.
Why this morphism is introduced depends on the next lemma.

(4-A.39) Lemma
$A_{\Sigma_1} = A_{\Sigma_2}$ if and only if $Mor(\Sigma_1, \Sigma_2) \neq \emptyset$.

[proof] This can be proved the same as in Matsuo [1977] and [1981].

(4-A.40) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold.
(1) If G_1 of Σ_1 is surjective, then dom $T_{12}^{min} = X_1$ holds, where dom T_{12}^{min} denotes the domain of T_{12}^{min}.
(2) If H_2 of Σ_2 is injective, then T_{12}^{max} is a partial function : $X_1 \to X_2$.

[proof] This can be proved the same as in Matsuo [1977] and [1981].

(4-A.41) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, then GrT_{12}^{max} is an invariant sub-product linear U-action of (X_1, F_1) and (X_2, F_2).

[proof] By the definition of $T_{12}^{max}, GrT_{12}^{max} = \{(x_1, x_2) \in X_1 \times X_2; H_1x_1 = H_2x_2\}$ holds. Let (x_1, x_2) and $(x_{1'}, x_{2'}) \in GrT_{12}^{max}$, i.e., $H_1x_1 = H_2x_2$ and $H_1x_{1'} = H_2x_{2'}$ hold. $H_1(x_1 + x_{1'}) = H_1x_1 + H_1x_{1'} = H_2x_2 + H_2x_{2'} = H_2(x_2 + x_{2'})$ hold. This implies $(x_1 + x_{1'}, x_2 + x_{2'}) \in GrT_{12}^{max}$. For $k \in K$ and $(x_1, x_2) \in GrT_{12}^{max}$, $(kx_1, kx_2) \in GrT_{12}^{max}$ holds. Moreover, for $u \in U$ and $(x_1, x_2) \in GrT_{12}\ max$, $H_1F_1(u)x_1 = S_l(u)H_1x_1 = S_l(u)H_2x_2 = H_2F_2(u)x_2$ hold. Hence, we obtain $(F_1(u)x_1, F_2(u)x_2) \in GrT_{12}^{max}$. Therefore, $GrT_{12}\ max \subseteq X_1 \times X_2$ is invariant under $F_1 \times F_2$. Therefore, $(GrT_{12}^{max}, F_1 \times F_2)$ is a linear U-action.

(4-A.42) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, G_1 be surjective and H_2 be injective, then $T_{12}^{min} = T_{12}^{max}$ holds and T_{12} is a Linear Representation System morphism : $\Sigma_1 \to \Sigma_2$ by setting $T_{12} = T_{12}^{min}$.

[proof] If G_1 is surjective and H_2 is injective, then Lemma (4-A.40) implies that $T_{12} \in Mor(\Sigma_1, \Sigma_2)$ is unique, $T_{12}G_1 = G_2$ and $H_2T_{12} = H_1$ hold. Owing to Lemma (4-A.41), T_{12} is a linear U-morphism : $(X_1, F_1) \to (X_2, F_2)$.

Remark: The uniqueness part of (sophisticated) Realization Theorem (4-A.37) for input response maps is proven by the canonicality of sophisticated Linear Representation Systems and Lemma (4-A.42).

4.5.B Finite Dimensionality

In this part, we will give proofs for theorems, Propositions and corollaries stated in section 4.2.

4.5.B.1 Pointed Finite Dimensional Linear U-Actions

In Appendix 4.5.A, the linear U-actions were introduced. In this section, we consider those whose state space is finite dimensional. Then it is shown that finite dimensional linear U-actions can be represented by matrix expressions.

(4-B.1) Definition
A linear U-action (X, F) of which X is finite (n) dimensional is said to be a finite dimensional (n dimensional) linear U-action.

In Appendix 4.5.A, we showed that an initial object of any pointed linear U-action $((X, F), x^0)$ is $((A(\Omega), S_r, \mathbf{e_1})$ and the quasi-reachability of $((X, F), x^0)$ implies a surjection of the corresponding linear input map G. In this section, we will give a criterion for being quasi-reachable of pointed finite dimensional linear U-actions. Introducing the quasi-reachable standard form, we show that it is a representative of pointed linear U-actions.

Let $((X, F), x^0)$ be a pointed linear U-action and G be the linear input map corresponding to an initial state x^0, namely, a linear U-morphism $:((A(\Omega), S_r) \to (X, F)$ which satisfies $G(\mathbf{e_1}) = x^0$.
Let $QR(i)$ be the linear hull of reachable set by input whose length is within i, i.e., $QR(i) :=\ll \{\sum \lambda_j x_j ; x_j = \phi_F(\omega_j)x^0,\ \omega_j \in \Omega_i\ \lambda_j \in K\} \gg$
. Where $\Omega_i := \{\omega \in \Omega\ ;\ |\omega| \leq i \in N\}$.
Then the following formula holds.
$QR(i+1) = QR(i) + \ll \{F(u)x; u \in U, x \in QR(i)\} \gg$.
Therefore, the following sequence can be obtained.
$QR(0) \subseteq QR(1) \subseteq \cdots QR(i) \subseteq \cdots \subseteq QR(\infty)$.
And $QR(n) = G(\ll \Omega_n \gg)$ holds. Where, $\ll \Omega_n \gg$ denotes the linear hull of Ω_n in $A(\Omega)$.

Moreover, let $G_l = G \cdot J_l$, where J_l is the canonical injection $: \ll \Omega_l \gg \to A(\Omega)$.
Then the above sequence can be rewritten as the following.
$\text{im } G_0 \subseteq \text{im } G_1 \subseteq \cdots \subseteq \text{im } G_i \subseteq \cdots \subseteq \text{im } G_\infty$.
Then we can obtain the next lemma easily.

(4-B.2) Lemma
If im G_{j-1} = im G_j for an integer $j \in N$ then im G_j = im G_{j+1}.

[proof] By the formula, im G_j = im $G_{j-1} + \{F(u)x; u \in U, x \in$ im $G_{j-1}\}$ holds. By assumption im G_{j-1} = im G_j , im G_{j+1} = im $G_{j-1} + \{F(u)x; u \in U$, $x \in$ im $G_{j-1}\}$ = im G_j holds.

(4-B.3) Lemma
For any pointed linear U-action $((K^n, F), x^0)$, then im G_{n-1} = im G always holds. Therefore, ((im $G_{n-1}, F), x^0)$ is a quasi-reachable pointed linear U-action.

[proof] This is a direct consequence of Lemma (4-B.2) and definition of quasi-reachability.

(4-B.4) Proposition
Let $((K^n, F), x^0)$ be a pointed linear U-action, then $((K^n, F), x^0)$ is quasi-reachable if and only if im $G_{n-1} = K^n$ holds.

[proof] The necessary and sufficient condition for being quasi-reachable of $((K^n, F), x^0)$ is that im $G = K^n$. By Lemma (4-B.3), this is equivalent to im $G_{n-1} = K^n$. Consequently, the proposition holds.

(4-B.5) Proposition
Let $((K^n, F), x^0)$ be a quasi-reachable pointed linear U-action, then im G_{j-1} is more than j dimensional for any integer $j (1 \leq j \leq n)$.

[proof] For any integer j, let's assume that there does not exist j linearly independent vectors in im G_{j-1}. And if im $G_{j-2} \subseteq$ im G_{j-1} holds, then the condition contradicts the nonexistence of j vectors. Hence, im G_{j-2} = im $G_{j-1} = \cdots =$ im G_∞ holds and im G_∞ has no more than j vectors. This contradicts the quasi-reachability of $((K^n, F), x^0)$.

(4-B.6) Proposition
Let $((K^n, F), x^0)$ be a pointed linear U-action. $((K^n, F), x^0)$ is quasi-reachable if and only if
rank $[x^0, F(u_1)x^0, F(u_2)x^0, \cdots, F(u_m)x^0, F(u_1)^2x^0,$

$F(u_1)F(u_2)x^0, F(u_1)F(u_3)x^0, \cdots, F(u_1)F(u_m)x^0, F(u_2)^2x^0, \cdots,$
$F(u_1)^{n-1}x^0, F(u_1)^{n-2}F(u_2)x^0, \cdots, F(u_m)^{n-1}x^0] = n$
holds.

[proof] This can be obtained by Proposition (4-B.4).

(4-B.7) Definition
Let $((K^n, F), x^0)$ be a quasi-reachable pointed linear U-action. If $((K^n, F), x^0)$ satisfies the following conditions, then it is said to be the quasi-reachable standard form:
1) $\mathbf{e_i} = \phi_{F_s}(\omega_i)\mathbf{e_1}$ hold for input sequences $\{\omega_i; 1 \leq i \leq n\}$.
2) $1 = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ hold.
3) $\phi_{F_s}(\omega)\mathbf{e_1} = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}$ holds for any input sequence ω such that $\omega_j < \omega < \omega_{j+1} (1 \leq j \leq n-1)$.

Remark: If $((K^n, F), x^0)$ is the quasi-reachable standard form, note that $x^0 = \mathbf{e_1}$.

(4-B.8) Proposition
For any quasi-reachable pointed linear U-action $((K^n, F), x^0)$, there uniquely exists the quasi-reachable standard form $((K^n, F), \mathbf{e_1})$ which is isomorphic to it.

[proof] We select the set of linearly n independent vectors $\{\phi_{F_s}(\omega_i)x^0; 1 \leq i \leq n\ ,\ \omega_i \in \Omega\}$ in the order of index value of Ω. Then the condition $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ holds by Proposition (4-B.5).
We introduce a linear operator $T : K^n \to K^n$ by setting $T\phi_F(\omega_i)x^0 = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_s(u) := TF(u)T^{-1}$ for any $u \in U$, then $F_s(u) \in K^{n \times n}$ and a collection $((K^n, F_s), \mathbf{e_1})$ is a pointed linear U-action. Since $T\phi_F(\omega_i)x^0 = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, the state $\mathbf{e_i}$ is a reachable state by input ω_i whose length is shorter than $i-1$. T is a pointed linear U-morphism : $((K^n, F), x^0) \to ((K^n, F_s), \mathbf{e_1})$. T preserves the linear independence and dependence. Therefore, $((K^n, F_s), \mathbf{e_1})$ is a quasi-reachable standard form.
Next, we can show its uniqueness comes from the selection of $\{\omega_i; 1 \leq i \leq n\}$.

Remark: There are many equivalence classes in the category of pointed linear U-actions, and this proposition says that the equivalence classes can be represented as quasi-reachable standard forms.

4.5.B.2 Finite Dimensional Linear U-Actions with a Readout Map

In Appendix 4.5.A, we showed that a final object of any linear U-action with a readout map $((X, F), h)$ is $((F(\Omega, Y), S_l), 1)$ and the distinguishability of $((X, F), h)$ implies an injection of the corresponding linear observation map H.

In this section, we will give a criterion for being distinguishable of finite dimensional linear U-actions with a readout. Introducing the distinguishable standard form, we show that it is a representative of linear U-actions with a readout map.

Let $((X, F), h)$ be a linear U-action with a readout map and H be the linear observation map corresponding to a readout map h, namely, a linear U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$ which satisfies $1H = h$.
Let $LO(i)$ be the linear hull of reachable set by output whose length is within i, i.e. $LO(i) := \{\sum \lambda_j x_j^*; x_j^* = h\phi_F(\omega_j),\ \lambda_j \in K, \omega_j \in \Omega_i\}$.
Where $\Omega_i := \{\omega \in \Omega; |\omega| \leq i\}$.
Then the following sequence holds.
$LO(0) \subseteq LO(1) \subseteq \cdots \subseteq LO(i) \subseteq \cdots \subseteq LO(\infty)$.

Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(\Omega, Y) \to F(\Omega_l, Y)$, and $F(\Omega_l, Y) := \{a \in F(\Omega, Y); a : \Omega_l \to Y\}$.
Then ker $H_l = LO(l)^0$ holds, i.e., ker $H_l = \{x \in X; hx = 0 \text{ for } h \in LO(l)\}$.
Moreover, ker $H = LO(\infty)^0$ holds.

(4-B.9) Lemma
For any linear U-action with a readout map $((K^n, F), h)$, $LO(n-1) = \ll h\phi_F(\Omega) \gg$ holds. Where $\ll h\phi_F(\Omega) \gg = \ll \{h\phi_F(\omega); \omega \in \Omega\} \gg$.

[proof] This can be obtained the same way as Lemma (4-B.3).

(4-B.10) Proposition
For any linear U-action with a readout map $((K^n, F), h)$, ((ker $H_{n-1}, F)$ is a sub linear U-action of (K^n, F) and $((K^n/_{\ker H_{n-1}}, \tilde{F}), \tilde{h})$ is a distinguishable linear U-action with a readout map.

[proof] Let H be the corresponding linear observation map to h. By Lemma (4-B.9), $LO(n-1) = \ll h\phi_F(\Omega) \gg$ holds. Therefore, ker $H_{n-1} =$ ker H holds. Because H is a linear U-morphism : $(K^n, F) \to (F(\Omega, Y), S_l)$,

$(\ker H_{n-1}, F)$ is a sub linear U-action of (K^n, F). Therefore, $((K^n/_{\ker\ H_{n-1}}, \tilde{F}), \tilde{h})$ can be introduced, and become a distinguishable linear U-action with a readout map.

(4-B.11) Proposition
Let $((K^n, F), h)$ be a linear U-action with a readout map. $((K^n, F), h)$ is distinguishable if and only if $LO(n-1) = K^n$ holds.

[proof] This can be obtained the same as Proposition (4-B.4).

(4-B.12) Proposition
If $((K^n, F), h)$ is distinguishable, then $LO(j-1)$ is more than j dimensional for any $j(1 \leq j \leq n)$.

[proof] This can be obtained the same as Proposition (4-B.5).

(4-B.13) Proposition
Let $((K^n, F), h)$ be a linear U-action with a readout map. $((K^n, F), h)$ is distinguishable if and only if
rank $[h^T, (hF(u_1))^T, EE, (hF(u_m))^T, (hF(u_1)^2)^T, (hF(u_1)F(u_2))^T, (hF(u_1)^{n-1})^T, \cdots, (hF(u_m)^{n-1})^T] = n$ holds.
Where T denotes the transpose of matrix.

[proof] This can be obtained the same as Proposition (4-B.6).

(4-B.14) Definition
Let $((K^n, F), h)$ be a distinguishable linear U-action with a readout map. If $((K^n, F), h)$ satisfies the following conditions, then it is said to be the distinguishable standard form:
1) $\mathbf{e_i}^T = h\phi_{F_s}(\omega_i)$ holds for input sequences $\{\omega_i; 1 \leq i \leq n\}$.
2) $1 = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ hold.
3) $h\phi_{F_s}(\omega) = \sum_{i=1}^{n} \alpha_i \mathbf{e_i}$ holds for any input sequence ω such that $\omega_j < \omega < \omega_{j+1}(1 \leq j \leq n-1)$.

Remark: If $((K^n, F), h)$ is the distinguishable standard form, note that $h = \mathbf{e_1}^T$.

(4-B.15) Propositon
For any distinguishable linear U-action with a readout map $((K^n, F), h)$, there uniquely exists the distinguishable standard form $((K^n, F_d), \mathbf{e_1}^T)$ which is isomorphic to it.

[proof] We select the set of linearly n independent vectors $\{h\phi_F(\omega_i); 1 \leq i \leq n, \omega_i \in \Omega\}$ in the order of index value of Ω. Then the condition $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ holds by Proposition (4-B.12). We introduce a linear operator $T : K^n \to K^n$ by setting $h\phi_F(\omega_i)T = \mathbf{e_i}^T$, for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_d(u) := TF(u)T^{-1}$ for any $u \in U$, then $F_d(u) \in K^{n\times n}$ and a collection $((K^n, F_d), \mathbf{e_1}^T)$ is a linear U-action with a readout map. T is a linear U-morphism with a readout map : $((K^n, F), h) \to ((K^n, F_d), \mathbf{e_1}^T)$. T preserves the linear independence and dependence. Therefore, $((K^n, F_d), \mathbf{e_1}^T)$ is the distinguishable standard form. Next, we can show the uniqueness of it come from the selection of $\{\omega_i; 1 \leq i \leq n\}$.

Remark: There are many equivalence classes in the category of linear U-actions with a readout map, and this proposition says that the equivalence class can be represented as the distinguishable standard forms.

4.5.B.3 Finite Dimensional Linear Representation Systems

This section is prepared for the proofs of Representation Theorems (4.14) and (4.16) for finite dimensional canonical Linear Representation Systems.

(4-B.16) Proof of Representation Theorem (4.14)
Note that a pointed linear U-action in the quasi-reachable standard system is the quasi-reachable standard form. Let $\sigma = ((K^n, F), x^0, h)$ be any finite dimensional canonical Linear Representation System. For the quasi-reachable standard form $((K^n, F_d), \mathbf{e_1}^T)$ and a pointed linear U-morphism $T : ((K^n, F), x^0) \to ((K^n, F_s), \mathbf{e_1})$ introduced in the proof of Theorem (4-B.8), let $h_s := h \cdot T^{-1}$. Then T is a Linear Representation System morphism : $((K^n, F), x^0, h) \to ((K^n, F_s), \mathbf{e_1}, h_s)$. And T is bijective and $((K^n, F_s), \mathbf{e_1}, h_s)$ is the only quasi-reachable standard system. By Corollary (4.6), the behaviors of $((K^n, F), x^0, h)$ and $((K^n, F_s), \mathbf{e_1}, h_s)$ are the same.

(4-B.17) Proof of Representation Theorem (4.16)
Note that a linear U-action with a readout map in the quasi-reachable standard system is the distinguishable standard form. Let $\sigma = ((K^n, F), x^0, h)$ be any finite dimensional canonical Linear Representation System. For the distinguishable standard form $((K^n, F_d), \mathbf{e_1}^T)$ and a linear U-morphism with readout map $T : ((K^n, F), h) \to ((K^n, F_d), \mathbf{e_1}^T)$ introduced in the proof of Theorem (4-B.15), let $x_d^0 := Tx\ 0$. Then T is a Linear Representation System morphism : $((K^n, F), x^0, h) \to ((K^n, F_d), x_d^0, \mathbf{e_1}^T)$. And T is bijective and $\sigma_d = ((K^n, F_d), x_d^0, \mathbf{e_1}^T)$ is the only distinguishable standard system. By Corollary (4.6), the behaviors of σ and σ_d are the same.

4.5.B.4 Existence Criterion for Linear Representation Systems

This section is prepared for the proofs of the theorem for existence criterion (4.18).
Let $G_l = G \cdot J_l$, where J_l is the canonical injection : $\ll \Omega_l \gg \to A(\Omega)$.
Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(\Omega, Y) \to F(\Omega_l, Y)$,

(4-B.18) Proof of Theorem (4.18)
Let A be the linear input/output map corresponding to input response map $a \in F(\Omega, Y)$. Obviously, im $A = \{S_l(\omega)a; \omega \in \Omega\}$. Let $A_l := A \cdot J_l$, and let a linear operator $A_{(l,m)} : \Omega_l \to F(\Omega_m, Y)$ be defined by setting $A_{(l,m)} := P_m \cdot A \cdot J_l$, then $A_{(l,m)}$ can be represented by a partial Hankel matrix $H^L_{a\ (l,m)}$ of the Hankel matrix H^L_a, where $H^L_{a\ (l,m)} = [a(\bar{\omega}|\omega)]$ for $\omega \in \Omega_l$ and $\bar{\omega} \in \Omega_m$.
First, we show 1) $\Longrightarrow$ 2). By Theorem (4.3) and Corollary (4-A.38), im A is n dimensional. If im $A_{n-1} \neq$ im A_n then the dimension of im A_n is $n+1$ or more by Lemma (4-B.2), therefore, im $A_{n-1} =$ im $A_n = \cdots =$ im A. Consequently, there exist n linearly independent vectors in $\{S_l(\omega)a; |\omega| \leq n-1$ for $\omega \in \Omega\}$, but not $n+1$ or more linearly independent vectors in it.
Secondly, we show 2) $\Longrightarrow$ 3). Since im $A_{n-1} =$ im A_n, im $A_{n-1} =$ im $A_n = \cdots =$ im A holds. Therefore, the dimension of im A_r is n for $r \leq n-1$. On the other hand, by Corollary (4-A.38) and Lemma (4-B.9), ker $P_s = 0$ for $s \leq n-1$. Consequently, the dimension of im $P_s \cdot A \cdot J_r$ is n. Therefore, the rank of partial Hankel matrix $H^L_{a\ (r,s)}$ corresponding to $P_s \cdot A \cdot J_r$ is n.
Lastly, we show 3) $\Longrightarrow$ 1). Since the rank of the Hankel matrix H^L_a is n, the range im A of the linear input/output map A corresponding to H^L_a is n dimensional. By im $A = \ll \{S_l(\omega)a; \omega \in \Omega\} \gg$ and Corollary (4-A.38), 1) is obtained.

4.5.B.5 Realization Procedure for Linear Representation Systems

This section is prepared for the proof of theorem for realization procedure (4.21).

(4-B.19) Proof of Theorem (4.21)
Let $R(a) := \{S_l(\omega)a; \omega \in \Omega\}$. By Theorem (4.3), $((\ll R(a) \gg, S_l), a, 1)$ is a canonical Linear Representation System that realizes $a \in F(\Omega, Y)$. The linearly independent vectors $\{S_l(\omega_i)a; 1 = \omega_1 < \omega_2 < \cdots < \omega_n, \omega_i \in \Omega$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)\}$ satisfies $\{S_l(\omega_i)a; 1 = \omega_1 < \omega_2 < \cdots < \omega_n,$ $\omega_i \in \Omega$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)\} = R(a)$. Let a linear map $T :$ $\ll R(a) \gg \rightarrow K^n$ be $T \cdot S_l(\omega_i)a = \mathbf{e_i}$ for any $i(1 \leq i \leq n)$. Then, by step 3), $h_s \cdot T = 1$ holds. And by step 4), $F_s(u) \cdot T = T \cdot F_s(u)$ holds for any $u \in U$. Consequently, T is bijective and a Linear Representation System morphism $: ((\ll R(a) \gg, S_l), a, 1) \rightarrow ((K^n, F_s), \mathbf{e_1}, h_s)$.
By Corollary (4.6), the behavior of $((K^n, F_s), \mathbf{e_1}, h_s)$ is a. It follows from the choice of $\{S_l(\omega_i)a; \omega_i \in \Omega$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)\}$ and the determination of map T that $((K^n, F_s), \mathbf{e_1}, h_s)$ is the quasi-reachable standard system.

4.5.C Partial Realization

In this appendix, we give proofs for theorems and propositions stated in section 4. See Appendices 4.5.A and 4.5.B for details of notions and notations.

4.5.C.1 Pointed Linear U-Actions

Set $\Omega_p := \{\omega \in \Omega; |\omega| \leq p$ for some $p \in N\}$, and $A(\Omega_p) := \{\sum_\omega \lambda(\omega)\mathbf{e}_\omega \in A(\Omega); \omega \in \Omega_p\}$, and let J_p be the canonical injection $\Omega_p \rightarrow A(\Omega)$.

(4-C.1) Definition
If a pointed linear U-action $((X,F),x^0)$ satisfies $X = \ll \{\phi_F(\omega)x^0;\ \omega \in \Omega_p\} \gg$, $((X,F),x^0)$ is called p-quasi reachable.

Remark: Note that $((X,F),x^0)$ is p-quasi-reachable if and only if $G_p := G \cdot J_p : A(\Omega_p) \to X$ is surjective. Where G is the linear input map : $(A(\Omega),S_r) \to (X,F)$ corresponding to $((X,F),x^0)$.

(4-C.2) Proposition
If a linear sub space S of $A(\Omega_p)$ satisfies the next two conditions, then there uniquely exists an ideal $\underline{S} \subseteq A(\Omega)$ such that $\underline{S} \cap A(\Omega_{p+1}) = S$ and $A(\Omega_{p+1})/S$ is isomorphic to $A(\Omega)/\underline{S}$. Moreover, a pointed linear U-action $((A(\Omega)/\underline{S},\tilde{S_r}),\mathbf{e_1}+\underline{S})$ is p-quasi-reachable.
Where $\tilde{S_r}$ is given by $\tilde{S_r}(\lambda+\underline{S}) = S_r\lambda+\underline{S}$.
condition 1: $\lambda \in A(\Omega_p) \cap S$ implies $S_r(u)\lambda \in S$ for any $u \in U$.
condition 2: There exist coefficients $\lambda(\omega_i) \in K$ such that $e_\omega - \sum_{\omega_i \in \Omega_p} \lambda(\omega_i)\mathbf{e}_{\omega_\mathbf{i}} \in S$ for any $\omega \in \Omega$, $|\omega| = p+1$.

[proof] Let $J_{(p,p+1)} : A(\Omega_p) \to A(\Omega_{p+1})$ be the canonical injection and $\pi_S : A(\Omega_{p+1}) \to A(\Omega_{p+1})/S$ be the canonical surjection. Then condition 2 implies that a composition map $\pi_S \cdot J_{(p,p+1)}$ is surjective. And condition 1 implies that $S_r(u)\lambda \in S$ holds for any $u \in U$ and $\lambda \in S$. Therefore, by setting $\tilde{S_r}(u)(\lambda+S) = S_r\lambda+S$ for any $\lambda \in A(\Omega_{p+1})$, we can uniquely define a map $\tilde{S_r} : U \to A(\Omega_{p+1})/S$. And $((A(\Omega_{p+1})/S,\tilde{S_r}),\mathbf{e_1}+S)$ is a pointed linear U-action and it is p-quasi-reachable. Then a linear input map $G : (A(\Omega),S_r) \to (A(\Omega_{p+1})/S,\tilde{S_r})$ corresponding to $((A(\Omega_{p+1})/S,\tilde{S_r}),\mathbf{e_1}+S)$ is uniquely determined by Proposition (4-A.27). Setting $G_{p+1} := G \cdot J_{p+1}$, $\ker G_{p+1} = S$ holds and $\underline{S} := \ker G$ satisfies $\underline{S} \cap A(\Omega_{p+1}) = S$. Since G is a linear input map, $\underline{S}$ is an invariant sub space under $S_r(u)$ for any $u \in U$. Moreover, the surjection of G implies that $((A(\Omega_{p+1})/S,\tilde{S_r}),\mathbf{e_1}+S)$ is isomorphic to $((A(\Omega)/\underline{S},\tilde{S_r}),\mathbf{e_1}+\underline{S})$. Therefore, $((A(\Omega)/\underline{S},\tilde{S_r}),\mathbf{e_1}+\underline{S})$ is p-quasi-reachable. The uniqueness of $\underline{S}$ is obtained by the uniqueness of $\tilde{S_r}$ and G.

4.5.C.2 Linear U-Actions with a Readout Map

Set $F(\Omega_q,Y) := \{\text{a function} : \Omega_q \to Y\}$, let P_q be the canonical surjection : $F(\Omega,Y) \to F(\Omega_q,Y); a \mapsto [;\omega \mapsto a(\omega)]$, and define $\underline{S_l}$ by setting $\underline{S_l}(\omega) : F(\Omega_q,Y) \to F(\Omega_{q-|\omega|},Y); a \mapsto \underline{S_l}(\omega)a[;\bar{\omega} \mapsto a(\bar{\omega}|\omega)]$.

(4-C.3) Definition
If a linear U-action with a readout map $((X,F),h)$ satisfies that $h\phi_F(\omega)x = 0$ implies $x=0$ for any $\omega \in \Omega_q$, it is called q-distinguishable.

Remark: Note that $((X,F),h)$ is q-distinguishable if and only if a linear map $H_q := P_q \cdot H$ is injective.
Where H is a linear observation map corresponding to $((X,F),h)$.

(4-C.4) Proposition
If a sub space Z of $F(\Omega_{q+1},Y)$ satisfies the next two conditions, then there uniquely exists a linear U-action (X,S_l) such that a map $P_{q|X} : X \to Z$ is isomorphic. Where $P_{q|X}$ is a restriction of the canonical surjection $P_q : F(\Omega,Y) \to F(\Omega_q,Y)$ to X. And a linear U-action with a readout map $((X,S_l,1)$ is q-distinguishable.
condition 3: A composition map $\pi \cdot j : Z \overset{j}{\to} F(\Omega_{q+1},Y) \overset{\pi}{\to} F(\Omega_q,Y)$ is injective.
condition 4: im $(\underline{S_l}(u) \cdot j) \subseteq$ im $(j \cdot \pi)$ holds in the sense of $F(\Omega_q,Y)$.
Where π is the canonical surjection.

[proof] By conditions 3 and 4, we can define $F(u)z = (\pi \cdot j)^{-1}\underline{S_l}(u) \cdot jz$ for any $u \in U$ and $z \in Z$. Then F is a map : $U \to L(Z)$. Hence, $((Z,F),1)$ is a distinguishable linear U-action with a readout map. Where 1 is a map : $Z \to Y; a \mapsto a(1)$. Injection of $\pi \cdot j$ implies that $((Z,F),1)$ is q-distinguishable. It follows that the linear observation map H corresponding to $((Z,F),1)$ is injective. Set $X :=$ im H, a map $H^{-1} : X \to Z$ is clearly the restriction of the map $P_q : F(\Omega,Y) \to F(\Omega_q,Y)$ to X.
An equation $1 = 1 \cdot H$ implies that $((X,S_1),1)$ is isomorphic to $((Z,F),1)$ in the sense of linear U-action with a readout map. Therefore, $((X,S_l),1)$ is q-distinguishable. A uniqueness of X is obtained by the uniqueness of F and H.

4.5.C.3 Partial Realization Problem

We can consider a partial linear input/output map $A_{(p,\underline{N}-p)} : A(\Omega_p) \to F(\Omega_{\underline{N}-p},Y)$ for $\underline{a} \in F(\Omega_{\underline{N}},Y)$ the same as the linear input/output map $A : (A(\Omega),S_r) \to (F(\Omega,Y),S_l)$ considered to $a \in F(\Omega,Y)$ in Appendix 4.5.A.

(4-C.5) Lemma

Let $A_{(p,\underline{N}-p)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then the following diagrams commute:

1)

$$\begin{array}{ccc} A(\Omega_p) & \xrightarrow{A_{(p,\underline{N}-p)}} & F(\Omega_{\underline{N}-p}, Y) \\ \underline{i} \downarrow & & \downarrow \pi \\ A(\Omega_{p+1}) & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(\Omega_{\underline{N}-p-1}, Y) \end{array}$$

Where $\underline{i}$ is canonical injection and π is canonical surjection.

2)

$$\begin{array}{ccc} A(\Omega_p) & \xrightarrow{A_{(p,\underline{N}-p)}} & F(\Omega_{\underline{N}-p}, Y) \\ S_r(u) \downarrow & & \downarrow \underline{S_l} \\ A(\Omega_{p+1}) & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(\Omega_{\underline{N}-p-1}, Y) \end{array}$$

[proof] These can be obtained by direct calculation.

(4-C.6) Proposition

Let $A_{(p_1,\underline{N}-p_1)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and p_2 be any integers such that $0 \leq p_2 \leq p_1 < \underline{N}$.
If $\text{im } A_{(p_2+1,\underline{N}-p_2-1)} = \text{im } A_{(p_2,\underline{N}-p_2-1)}$, then $\text{im } A_{(p_1,\underline{N}-p_1)} = \text{im } A_{(p_2,\underline{N}-p_1)}$ holds.

[proof] Note that this proposition holds if and only if $\text{im } A_{(p_2+1,\underline{N}-p_2-1)} = \text{im } A_{(p_2,\underline{N}-p_2-1)}$ implies $\text{im } A_{(p_2+1+n,\underline{N}-p_2-1-n)} = \text{im } A_{(p_2,\underline{N}-p_2-1-n)}$ holds for any non-negative integer n. Therefore, we prove the latter by the inductive method. When $n = 0$, it holds by assumption. Let's assume it holds for $n = k$, i.e., assume that $\text{im } A_{(p_2+1+k,\underline{N}-p_2-1-k)} = \text{im } A_{(p_2,\underline{N}-p_2-1-k)}$. Then for any $\bar{\omega} \in \Omega, |\bar{\omega}| = p_2 + 1 + k + 1$ given by $\bar{\omega} = u|\omega_1$. By as-

sumption, there exist $\omega_j \in \Omega_{p_2}, \alpha_j \in K$ and $m \in N(1 \leq j \leq m)$ such that $\underline{S_l}(\omega_1)\underline{a} = \sum_{j=1}^{m} \alpha_j \underline{S_l}(\omega_j)\underline{a}$ in sense of $F(\Omega_{\underline{N}-p_2-1-k}, Y)$. Therefore, $\underline{S_l}(\bar{\omega})\underline{a} = \underline{S_l}(u)\underline{S_l}(\omega_1) = \sum_{j=1}^{m} \alpha_i \underline{S_l}(u)\underline{S_l}(\omega_j)\underline{a} = \sum_{j=1}^{m} \alpha_i \underline{S_l}(u|\omega_j)\underline{a}$ hold. Therefore, im $A_{(p_2+1+k+1, \underline{N}-p_2-1-k-1)} =$ im $A_{(p_2+1, \underline{N}-p_2-1-k-1)}$ holds. On the other hand, im $A_{(p_2+1, \underline{N}-p_2-1)} =$ im $A_{(p_2, \underline{N}-p_2-1)}$ is equivalent to im $A_{(p_2+1,j)} =$ im $A_{(p_2,j)}$ for any $j \leq \underline{N} - p_2 - 1$.
Therefore, im $A_{(p_2+1+k+1, \underline{N}-p_2-1-k-1)} =$ im $A_{(p_2, \underline{N}-p_2-1-k-1)}$ holds. The condition equation holds for $n = k + 1$.

(4-C.7) Proposition
Let $A_{(\, , \,)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. For p_1 and p_2 be any integers such that $0 \leq p_2 < p_1 < \underline{N}$.
If ker $A_{(p_1, \underline{N}-p_1)} =$ ker $A_{(p_1, \underline{N}-p_1-1)}$ hold, then ker $A_{(p_2, \underline{N}-p_1-1)}$ $=$ ker $A_{(p_2, \underline{N}-p_2)}$ holds.

[proof] Note that this proposition holds if and only if ker $A_{(p_1, \underline{N}-p_1)} =$ ker $A_{(p_1, \underline{N}-p_1-1)}$ implies ker $A_{(p_1-n, \underline{N}-p_1-1)} =$ ker $A_{(p_1-n, \underline{N}-p_1+n)}$ for any n in $0 \leq n \leq p_1$. Therefore, we prove the latter by the inductive method. When $n = 0$, it holds by assumption. Let's assume that it holds for $n = k$, i.e., assume that ker $A_{(p_1-k, \underline{N}-p_1-1)} =$ ker $A_{(p_1-k, \underline{N}-p_1+k)}$. Then, for any $u \in U$ and $\bar{\omega} \in \Omega, |\bar{\omega}| = \underline{N} - p_1 + k + 1$, let $\bar{\omega} = \bar{\omega}_1|u$. By assumption, there exist $\omega_j \in \Omega_{\underline{N}-p_1-1}$, $\alpha_j \in K$ and $m \in N(1 \leq j \leq m)$ such that $\underline{a}(\bar{\omega}|\omega) = \underline{a}(\bar{\omega_1}|u|\omega) = \sum_{j=1}^{m} \alpha_j \underline{a}(\omega_j|u|\omega)$. Since $\omega_j|u \in \Omega_{N-p_1}$, ker $A_{(p_1-k-1, \underline{N}-p_1)} =$ ker $A_{(p_1-k-1, \underline{N}-p_1+k+1)}$ holds. On the other hand, if we note that ker $A_{(p_1, \underline{N}-p_1)} =$ ker $A_{(p_1, \underline{N}-p_1-1)}$ is equivalent to ker $A_{(i, N-p_1)} =$ ker $A_{(i, \underline{N}-p_1-1)}$ for any i in $0 \leq i \leq p_1$, ker $A_{(p_1-k-1, \underline{N}-p_1-1)}$ $=$ ker $A_{(p_1-k-1, \underline{N}-p_1+k+1)}$ holds. Therefore, the condition's equation holds for $n = k + 1$.

(4-C.8) Lemma
For a partial linear input/output map $A_{(\, , \,)}$ corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and a Linear Representation System $\sigma = ((X, F), x^0, h)$, the next matters hold. Where $G_p := G \cdot J_p$, $H_q := P_q \cdot H$ for the linear input map G corresponding to x^0 and the linear output map H corresponding to h. $A_{p,q} := H_q \cdot J_p$.
1) σ is a partial realization of $\underline{a}$ if and only if the following figure commutes for any p such that $0 \leq p < \underline{N}$.
2) σ is a natural partial realization of $\underline{a}$ if and only if the following figure commutes, G_p is surjective and $H_{\underline{N}-p-1}$ is injective for some p such that $0 \leq p < \underline{N}$.

$$\begin{array}{ccccc}
A(\Omega_p) & \xrightarrow{G_p} & X & \xrightarrow{H_{\underline{N}-p}} & F(\Omega_{\underline{N}-p}, Y) \\
{\scriptstyle S_r(u)}\downarrow & & \downarrow{\scriptstyle F(u)} & & \downarrow{\scriptstyle \underline{S_l}} \\
A(\Omega_{p+1}) & \xrightarrow{G_{p+1}} & X & \xrightarrow{H_{\underline{N}-p-1}} & F(\Omega_{\underline{N}-p-1}, Y)
\end{array}$$

[proof] These can be obtained by definition of the partial and natural partial realization.

(4-C.9) Proof of Theorem (4.26)
We prove the theorem by rewriting the conditions of partial Hankel matrix in Theorem (4.26) to a partial linear input/output map $A_{(\ , \ )}$ corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. By using Proposition (4-C.6) and (4-C.7), the conditions of Hankel matrix can be equivalently changed to the following equations (1) and (2):

(1) im $A_{(p,\underline{N}-p-1)} =$ im $A_{(p,\underline{N}-p-1)}$
(2) ker $A_{(p,\underline{N}-p)} =$ ker $A_{(p,\underline{N}-p-1)}$
Therefore, we will prove the theorem by using (1) and (2).

First, we show that the above equations (1) and (2) are necessary. Let $\sigma = ((X,F), x^0, h)$ be a natural partial realization of $\underline{a} \in F(\Omega_N, Y)$, then σ is p-quasi-reachable, and q-distinguishable for some p and q such that $p+q < N$. Let G be the linear input map corresponding to x^0 and H be the linear output map corresponding to h, and let $p \leq p'$ and $q \leq q'$, then $G_{p'} := G \cdot J_{p'}$ is onto, $H_{q'} := P_{q'} \cdot H$ is one-to-one. Therefore, $A_{(p',q')} := H_{q'} \cdot J_{p'}$ satisfies equations (1) and (2).
Next, we show that the equations (1) and (2) are sufficient. Set $S :=$ ker $A_{(p+1,\underline{N}-p-1)}$ and $Z :=$ im $A_{(p,\underline{N}-p)}$. Then equation (2) implies that a composition map $\pi \cdot j : Z \overset{j}{\to} F(\Omega_{\underline{N}-p}, Y) \overset{\pi}{\to} F(\Omega_{\underline{N}-p-1}, Y)$ is injective. Where π and j are the same as in Proposition (4-C.4). Therefore, Z satisfies condition 3 in Proposition (4-C.4). Equation (1) implies that there exist $\lambda(\omega_i)\mathbf{e}_{\omega_\mathbf{i}}$ such that $A_{(p+1,\underline{N}-p-1)}(\mathbf{e}_\omega) = A_{(p,\underline{N}-p-1)}(\sum_i \lambda(\omega_i)\mathbf{e}_{\omega_\mathbf{i}}$ for any $\omega \in \Omega$ and $|\omega| = p+1$.

By Lemma (4-C.5), we obtain that $A_{(p+1,\underline{N}-p-1)}(\mathbf{e}_\omega - \sum_i \lambda(\omega_i)\mathbf{e}_{\omega_\mathbf{i}}) = 0$, and $\mathbf{e}_\omega - \sum_i \lambda(\omega_i)\mathbf{e}_{\omega_\mathbf{i}} \in S$ holds. This implies that S satisfies condition 2 in Proposition (4-C.2).
Let $\underline{j}$ be the canonical injection $:A_{(p,\underline{N}-p-1)} \to F(\Omega_{\underline{N}-p-1}, Y)$ and π is the same as in Proposition (4-C.4), $B := (\underline{j})^{-1} \cdot \pi \cdot j : Z \to \text{im } A_{(p,\underline{N}-p-1)}$ is a bijective linear map by (2) in Proposition (4-C.2). When we consider the bijective linear map $A^b := A^b_{(p+1,\underline{N}-p-1)} : A(\Omega_{p+1})/S \to \text{im } A_{(p+1,\underline{N}-p-1)}$ associated with $A_{(p+1,\underline{N}-p-1)} : A(\Omega_{p+1}) \to F(\Omega_{\underline{N}-p-1}, Y)$, equation (2) implies that a linear map $B^{-1} \cdot A^b$ is a bijective linear map $:A(\Omega_{p+1})/S \to Z$. For any $\lambda \in A(\Omega_p) \cap S$, $A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by injection of $B^{-1} \cdot A^b$. Therefore, $A_{(p+1,\underline{N}-p-1)}(S_r(u)\lambda) = \underline{S_l}(u)A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by using 2) in Lemma (4-C.5) for any $u \in U$. This implies that $S_r(u)\lambda \in S$. Therefore, S satisfies condition 1 in Proposition (4-C.2). Then Proposition (4-C.2) implies that a pointed linear U-action $((A(\Omega_{p+1})/S, \tilde{S}_r), \mathbf{e_1} + S)$ is p-quasi-reachable. Here, equation (1) implies that there exists $x \in \text{im } A_{(p,\underline{N}-p-1)}$ such that $\underline{j}(x) = \underline{S_l} \cdot j(z)$ for any $z \in Z$ and $u \in U$. Moreover, by surjection of B, there exists $z' \in Z$ such that $B(z') = x$. Therefore, $\underline{S_l} \cdot \underline{j}(z) = j(x) = \underline{j} \cdot B(z') = \pi \cdot j(z')$, which implies that im $(\underline{S_l}(u) \cdot j) \subseteq$ im $(\pi \cdot j)$. It follows that Z satisfies condition 4 in Proposition (4-C.4) and $((Z, F), 1)$ is $(\underline{N} - p - 1)$-distinguishable. We can also show that $B^{-1} \cdot A^b$ is a linear U-morphism : $A(\Omega_{p+1})/S, \tilde{S}_r) \to (Z, F)$, and that a Linear Representation System $\sigma_1 = ((A(\Omega_{p+1})/S, \tilde{S}_r), e_1 + S, 1 \cdot B^{-1} \cdot A^b)$ is isomorphic to a Linear Representation System $\sigma_2 = ((Z, F), B^{-1} \cdot A^b(\mathbf{e_1} + S), 1)$. It follows that σ_1 and σ_2 are the natural partial realizations of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Therefore, there exist the natural partial realizations of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$.

(4-C.10) Lemma
Two canonical Linear Representation Systems are isomorphic if and only if their behavior is the same.

[proof] This can be obtained from Theorem (4.5) and Corollary (4.6).

(4-C.11) Proof of Theorem (4.27)
Let $A_{(\ ,\ )}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. In order to prove necessity, we assume existence of the natural partial realization of $\underline{a}$. Let Theorem (4.26) hold for integers p and p' that are different. Namely,

(1) im $A_{(p,\underline{N}-p-1)} = \text{im } A_{(p+1,\underline{N}-p-1)}$
(2) ker $A_{(p,\underline{N}-p)} = \text{ker } A_{(p,\underline{N}-p-1)}$
(3) im $A_{(p',\underline{N}-p'-1)} = \text{im } A_{(p'+1,\underline{N}-p'-1)}$
(4) ker $A_{(p',\underline{N}-p')} = \text{ker } A_{(p',\underline{N}-p'-1)}$
Then Propositions (4-C.6) and (4-C.7) imply that the dimension of $Z = \text{im } A_{(p,\underline{N}-p-1)}$ is equal to one of $Z' = \text{im } A_{(p',\underline{N}-p'-1)}$. Let σ and σ' be the natural partial realizations of $\underline{a}$ whose state spaces are Z and Z' respectively and which can be obtained by the same procedure as in (4-C.9). Then σ is clearly isomorphic to σ' and the behavior of σ is equal to one of σ' by Lemma (4-C.10). This implies that the behavior of the natural partial realization is always the same regardless of different integers p and p'. Therefore, the natural partial realization of $\underline{a}$ is unique modulo isomorphism by Lemma (4-C.10).
Next, we show sufficiency by the contrapositive. We assume that there does not exist a natural partial realization of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then minimum dimensional partial realization σ of $\underline{a}$ is p-quasi-reachable and q-distinguishable for $p + q \geq \underline{N}$. It cannot be quasi-reachable within $p - 1$ and not be distinguishable within $q - 1$. Then, there exists a state x in σ such that x can be first reachable by a input ω with length p. The remaining data of $F(\Omega_{\underline{N}-p-1}, Y)$ can't determine a new state $F(u)x$ for any $u \in U$, because of $\underline{N} - p - 1 < q$. Therefore, we can't determine the transition matrix $F(u)$ uniquely by q-distinguishability. This implies that the minimum dimensional realization of $\underline{a}$ is not unique.

(4-C.12) Proof of Theorem (4.28)
Let's consider the natural partial realization $\sigma_2 = ((Z, F), B^{-1} \cdot A^b(e_1 + S), 1)$ of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ given in (4-C.9). Then we can obtain the quasi-reachable standard system $\sigma_s = ((K^n, F_s), \mathbf{e_1}, h_s)$ from σ_2 in the same manner as theorem for a realization procedure (4.21).

5 Affine Dynamical Systems

Let the output value's set Y be any linear space over the field K. The realization problem for Affine Dynamical Systems can be stated as the following: [For any input response (equivalently, any input output map with causality), there exist at least two Affine Dynamical Systems which realize (faithfully describe) it. Let σ_1 and σ_2 be Affine Dynamical Systems that have the same behavior, then σ_1 is isomorphic to σ_2 in the sense of Affine Dynamical Systems.]

T. Matsuo [1981] has obtained the theorem for Affine Dynamical Systems of continuous-time system.

Here, to obtain the realization theorem for Affine Dynamical Systems of discrete-time, we rewrite one of continuous-time to one of discrete-time by using special features of discrete-time. Therefore, we introduce Affine Dynamical Systems to be proper for discrete-time. The solution of the realization problem of any general non linear system (equivalently, any input response map) is given by the Affine Dynamical Systems.

In Appendix 5.4, we show how Affine Dynamical Systems of discrete time systems are rewritten from ones of continuous-time system established by Matsuo. Moreover, we prove Realization Theorem (5.15) by rewriting Affine Dynamical Systems to sophisticated Affine Dynamical Systems in Appendix 5.4.A.

5.1 Realization Theorem of Affine Dynamical Systems

(5.1) Definition

A system given by the following equations is written as a collection $\sigma^A = ((X^A, F^A), x^0, h^A)$ and it is said to be an Affine Dynamical System.

$$\begin{cases} x(t+1) &= F^A(\omega(t+1))x(t) \\ x(0) &= x^0 \\ \gamma(t) &= h^A x(t) \end{cases}$$

for any $t \in N$, where x(t) $\in X^A$ and $\gamma(t) \in Y^A$.

Where X^A is an affine space that may be called a state space, F^A is an affine map $: U \to AM(X^A); u \mapsto F^A(u)$, an initial state $x^0 \in X^A$ and $h^A : X^A \to Y^A$ is an affine map. And $Y^A = (Y, Y)$. See Appendix 5.4 for details.

The equation $x(t+1) = F^A(\omega(t+1))x(t)$ in the Affine Dynamical System σ^A may be said to be an affine U-action (X^A, F^A), and x^0 is called an initial state. The input response map $a_{\sigma^A} : \Omega \to Y^A; \omega \mapsto h^A \phi_{F^A}(\omega)x^0$ is said

T. Matsuo, Y. Hasegawa: Realization Theory of Discrete-Time Dynamical Systems, LNCIS 296, pp. 79–112, 2003.

to be the behavior of σ^A. For an input response map $a \in F(\Omega, Y)$, σ^A that satisfies $a_{\sigma^A} = a$ is called a realization of a.
Where $\phi_{F^A}(\omega) = F^A(\omega(|\omega|))F^A(\omega(|\omega|-1))\cdots F^A(\omega(1))$.

An Affine Dynamical System σ^A is said to be quasi-reachable if the smallest affine space that contains the reachable set $\{\phi_{F^A}(\omega)x^0; \omega \in \Omega\}$ is equal to X^A and an Affine Dynamical System σ^A is called distinguishable if $h^A\phi_{F^A}(\omega)x_1 = h^A\phi_{F^A}(\omega)x_2$ for any $\omega \in \Omega$ implies $x_1 = x_2$.

An Affine Dynamical System σ^A is called to be canonical if σ^A is quasi-reachable and distinguishable.

Remark 1: The $x(t)$ in the system equation of σ^A is the state that produces output values of a at the time t, namely the state x(t) and an affine map $h^A : X^A \to Y^A$ generates the output value $a_{\sigma^A}(t)$ at the time t.
Remark 2: It is meant for σ^A to be a faithful model for the input response map a such that σ^A realizes a.
Remark 3: Notice that a canonical Affine Dynamical System:
$\sigma^A = ((X^A, F^A), x^0, h^A)$ is a system which has the most reduced state set X^A among systems that have the behavior a_{σ^A} (see Proposition (5-A.16), Remark in Definition (5-A.17), Definition (5-A.21), Proposition (5-A.24), (5-A.25), Corollary (5-A.30) and Proposition (5-A.34)).

(5.2) Proposition
Let $X^A = (X, \mathbf{X})$ be an affine space. For any map $F^A := (F, \mathbf{F}) : U \to AM(X^A)$, there exists uniquely a map $g : U \to \mathbf{X}$ such that $F^A(u)(x^0 + \mathbf{x}) = x^0 + \mathbf{F}(u)\mathbf{x} + g(u)$, for any $x^0 \in X$, $\mathbf{x} \in \mathbf{X}$ and $u \in U$.

[proof] This can be obtained by Definition (5-A.1) and (5-A.2) in Appendix 5.4.

(5.3) Definition
Let $\sigma_1^A = ((X_1^A, F_1^A), x_1^0, h_1^A)$ and $\sigma_2^A = ((X_2^A, F_2^A), x_2^0, h_2^A)$ be Affine Dynamical Systems. If an affine map $T^A : X_1^A \to X_2^A$ satisfies $T^A F_1^A(u) = F_2^A(u) T^A$ for any $u \in U$, $T^A x_1^0 = x_2^0$ and $h_1^A = h_2^A T^A$, then T^A is said to be an Affine Dynamical System morphism $T^A : \sigma_1^A \to \sigma_2^A$. If the T^A is bijective then T^A is said to be an isomorphism.

(5.4) Corollary
Let σ_1^A and σ_2^A be Affine Dynamical Systems and $T^A : \sigma_1^A \to \sigma_2^A$ be an Affine Dynamical System morphism. Then $a_{\sigma_1^A} = a_{\sigma_2^A}$ holds.

[proof] This can be proved by direct calculation of behavior's definition in Definition (5.1).

(5.5) Proposition
An Affine Dynamical System $\sigma^A = ((X^A, F^A), x^0, h^A)$ can be rewritten by the following equations:

$$\begin{cases} x(0) & = & x^0 \\ x(t+1) & = & \mathbf{x}(t+1) + x^0 \\ \mathbf{x}(t+1) + x^0 & = & \mathbf{F}(\omega(t+1))x(t) + g(\omega(t+1)) + x^0 \\ \gamma(t) & = & hx^0 + \mathbf{hx}(t) \end{cases}$$

for any $t \in N$, where $x(t) \in X$, $\mathbf{x}(t) \in \mathbf{X}$ and $\gamma(t) \in Y^A$. And $\mathbf{F}(\omega(t))$ is the linear operator induced by $F^A(\omega(t))$, $\mathbf{h}$ is the linear operator induced by h^A. See Appendix 5.4 for induced operators.
Namely, for any Affine Dynamical System $\sigma^A = ((X^A, F^A), x^0, h^A)$, σ^A is isomorphic to the system given by the above equations in the sense of an Affine Dynamical System.

[proof] We can obtain this by Proposition (5.2) and Definition (5.1).

(5.6) Definition
According to Proposition (5.5), any Affine Dynamical System $\sigma^A = ((X^A, F^A), x^0, h^A)$ given in Definition (5.1) may be rewritten by the equations given by Proposition (5.5). If the latter equations decompose into a part of set and a part of linear space, then $\sigma^A = ((X^A, F^A), x^0, h^A)$ may be written by $\sigma^A = ((X, x^0, h^0), (\mathbf{X}, \mathbf{F}), g, \mathbf{h})$.
Where $h^0 = hx^0$, $\mathbf{F}$ and $\mathbf{h}$ are the linear operators induced by F^A and h^A respectively.

In order to clarify structures of Affine Dynamical Systems, we will define the following one.

(5.7) Definition
Let $\mathbf{X}$ be a linear space and $\mathbf{F}$ be a map : $U \to L(\mathbf{X})$, then a pair $(\mathbf{X}, \mathbf{F})$ is said to be a linear U-action. Let $(\mathbf{X_1}, \mathbf{F_1})$ and $(\mathbf{X_1}, \mathbf{F_2})$ be linear U-actions, a linear operator $T : \mathbf{X_1} \to \mathbf{X_2}$ that satisfies $\mathbf{TF_1}(u) = \mathbf{F_2}(u)\mathbf{T}$ is said to be a linear U-morphism : $(\mathbf{X_1}, \mathbf{F_1}) \to (\mathbf{X_2}, \mathbf{F_2})$.

(5.8) Proposition
A morphism f^A is an affine morphism f^A
$: \sigma_1^A = ((X_1, x_1^0, h_1^0), (\mathbf{X_1}, \mathbf{F_1}), g_1, \mathbf{h_1}) \to \sigma_2^A = ((X_2, x_2^0, h_2^0), (\mathbf{X_2}, \mathbf{F_2}), g_2, \mathbf{h_2})$
if and only if the following five conditions hold:
1) $\mathbf{f}$ is a linear U-morphism : $(\mathbf{X_1}, \mathbf{F_2}) \to (\mathbf{X_2}, \mathbf{F_2})$.
2) $f(x_1^0) = x_2^0$.
3) $\mathbf{f} g_1 = g_2$.
4) $\mathbf{h_1} = \mathbf{h_2}\mathbf{f}$.
5) $f h_1^0 = h_2^0$.
Where $f^A = (f, \mathbf{f})$.

[proof] This can be easily obtained by Definition (5.3), Proposition (5.5) and Definition (5.7).

(5.9) Proposition
Any Affine Dynamical System $\sigma^A = ((X^A, F^A), x^0, h^A)$ is isomorphic to $\sigma_n^A = ((X, 0, h^0), (\mathbf{X}, \mathbf{F}), g, \mathbf{h})$ whose initial state is zero in the linear space $\mathbf{X}$ in place of an initial x^0 of the set X.

[proof] If we consider an affine map $f^A = (f, I) : (X, \mathbf{X}) \to (\mathbf{X}, \mathbf{X})$ such that $f : X \to \mathbf{X}; x^0 \mapsto f(x^0) = 0$, then σ^A is isomorphic to σ_n^A.

Therefore, without loss of generality, we can define the Affine Dynamical System as the following:

(5.10) Definition
The system given by the following equations is said to be an Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$. And X is a linear space over the field K.

$$\begin{cases} x(0) & = & 0 \\ x(t+1) & = & F(\omega(t+1))x(t) + g(\omega(t+1)) \\ \gamma(t) & = & h^0 + hx(t) \end{cases}$$

for any $t \in N$, where $x(t) \in X$ and $\gamma(t) \in Y$. And $F(\omega(t+1)) \in L(X)$.
For the newly revised Affine Dynamical Systems, we can rewrite Proposition (5.9) to the following proposition to be more proper for them.

(5.11) Proposition
Let $\sigma_1 = ((X_1, F_1), g_1, h_1, h^0)$ and $\sigma_2 = ((X_2, F_2), g_2, h_2, h^0)$ be Affine Dynamical Systems. A morphism f is an affine dynamical morphism f : $\sigma_1 \to \sigma_2$ if and only if f is a linear map : $X_1 \to X_2$ that satisfies $fF_1(u) = F_2(u)f$, $fg_1 = g_2$ and $h_1 = h_2 f$.

[proof] This can be obtained easily.

(5.12) Example
Let $\Omega^+ := \Omega \backslash \mathit{1}$ and $V(\Omega^+) := \{\lambda = \sum_{\omega\in\Omega^+} \lambda(\omega)\mathbf{e}_\omega(\text{finite sum})\, ; \lambda(\omega) \in K\}$. Where $\mathbf{e}_\omega(\bar{\omega}) = 1$ for $\omega = \bar{\omega}$ and $\mathbf{e}_\omega(\bar{\omega}) = 0$ for $\omega \neq \omega$.
Let ψ be a map : $U \to L(V(\Omega^+)); u \mapsto \psi(u)[; \mathbf{e}_\omega \mapsto \mathbf{e}_{\mathbf{u}|\omega} - \mathbf{e}_{\mathbf{u}}]$.
And let a map $e : U \to V(\Omega^+); u \mapsto \mathbf{e}_{\mathbf{u}}$, where $e(\mathit{1}) = 0$. And we consider a linear map $a_l : V(\Omega^+) \to Y; \mathbf{e}_\omega \mapsto a(\omega) - a(\mathit{1})$ for any input response map $a \in F(\Omega, Y)$. Then $((V(\Omega^+), \psi), e, a_l, a(\mathit{1}))$ is a quasi-reachable Affine Dynamical System that realizes $a \in F(\Omega, Y)$.

(5.13) Example
Let $a \in F(\Omega, Y)$ be any input response map and S_l be defined by $S_l(u)a : \Omega \to Y; \omega \mapsto a(\omega|u)$. Then $S_l(u) \in L(F(\Omega, Y)$ for any $u \in U$. Let a map $\xi : U \to F(\Omega, Y)$ be $u \mapsto \xi(u)[; \omega \mapsto a(\omega|u) - a(\omega)]$. And let 1 be a linear map : $F(\Omega, Y) \to Y; a \mapsto a(\mathit{1})$. Then $((F(\Omega, Y), S_l), \xi, 1, a(\mathit{1}))$ is a distinguishable Affine Dynamical System that realizes $a \in F(\Omega, Y)$.

Remark: Examples (5.12) and (5.13) imply that there exist many Affine Dynamical Systems that realize a given input response map. However, there is no relation between them. Therefore, we introduce the canonical Affine Dynamical Systems, and we will make a clear relation between them.

(5.14) Theorem
For any input response $a \in F(\Omega, Y)$, there exist the following two canonical Affine Dynamical Systems that realize it.
1) $((V(\Omega^+)/_{=a}, \tilde{\psi}), [e_1], [\mathit{1}], \tilde{a}_l, a(\mathit{1}))$.
Where $V(\Omega^+)/_{=a}$ is a quotient space derived by equivalence relation:
$\sum_\omega \lambda(\omega)\mathbf{e}_\omega = \sum_{\bar{\omega}} \lambda(\bar{\omega})\mathbf{e}_{\bar{\omega}} \iff \sum_\omega \lambda(\omega)(a(\omega) - a(\mathit{1})) = \sum_{\bar{\omega}} \lambda(\bar{\omega})(a(\bar{\omega}) - a(\mathit{1}))$.
$\tilde{\psi}$ is given by a map :$U \to L(V(\Omega^+)/_{=a}); u \mapsto \tilde{\psi}(u)[; \lambda \mapsto \sum_\omega \lambda(\omega)(\mathbf{e}_{\mathbf{u}|\omega} - \mathbf{e}_{\mathbf{u}})$, and $\tilde{a}_l$ is given by $\tilde{a}_l : V(\Omega^+)/_{=a} \to Y; [\lambda] \mapsto \tilde{a}_l([\lambda]) = \sum_\omega \lambda(\omega)(a(\omega) - a(\mathit{1}))$.
2) $((\ll S_l(\Omega)a - a \gg, S_l), \xi, 1, a(\mathit{1}))$.

Where $S_l(\Omega)a - a = \{S_l(\omega)a - a; \omega \in \Omega\}$ and $\ll S_l(\Omega)a - a \gg$ denotes the smallest linear space which contains $S_l(\Omega)a - a$.

[proof] See Examples (5.12) and (5.13), Proposition (5-A.34) and Corollary (5-A.36).

We conclude that there exist the canonical Affine Dynamical Systems that realize any input response map in Theorem (5.14). Next, we will insist the uniqueness of the systems that have the same behavior.

(5.15) Realization Theorem
For any input response map $a \in F(\Omega, Y)$, there exist at least two canonical Affine Dynamical Systems that realize it.
Let $\sigma_1 = ((X_1, F_1), g_1, h_1, h^0)$ and $\sigma_2 = ((X_2, F_2), g_2, h_2, h^0)$ be canonical Affine Dynamical Systems that realize any $a \in F(\Omega, Y)$, then there exists uniquely an isomorphism $T : \sigma_1 \to \sigma_2$.

[proof] See Propositions (5-A.18), (5-A.25), (5-A.34), Remark in Proposition (5-A.16) and Remark in Lemma (5-A.40).

5.2 Finite Dimensional Affine Dynamical Systems

Based on Realization Theorem (5.15), we clarify the finite-dimensionality of the systems. Therefore, we obtain the same results as obtained in the linear systems by R. E. Kalman. It is intended that the finite dimensionality of Affine Dynamical Systems are fundamentally characterized.

Firstly, we assume that the set U of input's values is finite, and we show that the assumption of finiteness is not so special. Namely, Affine Dynamical Systems with the assumption include biaffine systems as a subclass. Biaffine systems were discussed by Tarn and Nonoyama [1979].

The following results are obtained for the systems. It is given a criterion for being canonical of finite dimensional Affine Dynamical Systems. We give a criterion for the behavior of finite dimensional Affine Dynamical Systems. The companion form for canonical finite-dimensional Affine Dynamical Systems is also given. Moreover, a procedure to obtain the companion form from a given input/output map is obtained.

Lastly, as an example of the procedure, we treat a typical hysteresis characterestic. Therefore, it is obvious that this theory is the extension of the linear system theory established by Kalman et al to the non-linear case.

An Affine Dynamical System is different from a state-affine system in Sontag [1979a]. Our system is introduced on the basis of Theorem (2.6) and Definition (2.7) in Chapter 2, which is the representation theorem for any input/output map with causality. Hence, our systems are more general than state-affine systems.

In order to obtain meaningful and useful results, we introduce finite dimensional Affine Dynamical Systems. If the state space X of an Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ is finite dimensional (n-dimensional), then σ is said to be a finite dimensional (n-dimensional) Affine Dynamical System.

There is the following fact about n dimensional linear space in Halmos [1958].
Fact: [Every n dimensional linear space over the field K is isomorphic to K^n. Moreover, every linear operator from K^n to K^m is isomorphic to a matrix $F \in K^{m\times n}$.]
Therefore, without loss of generality, an n dimensional Affine Dynamicai System can be represented by $\sigma = ((X, F), g, h, h^0)$.
Where, F is a map : $U \to K^{n\times n}$, g is a map : $U \to K^n$ and $h \in K^{p\times n}$ and $h^0 \in K^p$.

According to the above discussion, we can treat an n-dimensional Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ which is easily embodied by computer programs or electrical circuits.

From now on, we assume that the set U of input's values is finite. Let $U = \{u_1, u_2, \cdots, u_m\}$. Now, we show that the assumption is not so special.

(5.16) Biaffine Systems
We will consider the following system:

$$\begin{cases} x(t+1) & = & (A + \sum_{i=1}^m N_i \cdot \omega_i(t+1))x(t) + \sum_{i=1}^m \mathbf{b}_i \cdot \omega_i(t+1) + \mathbf{a} \\ x(0) & = & 0 \\ \gamma(t) & = & h^0 + hx(t) \end{cases}$$

$\omega_i(t) \in \mathbf{R}$, x(t), $\mathbf{b}_i$ and $\mathbf{a} \in \mathbf{R}^n$, $N_i \in \mathbf{R}^{n\times n}$ and $\gamma(t) \in Y$.

Transfering the time in input, we will conclude that the above system is a biaffine system treated in Tarn and Nonoyama [1979].
Where maps $\tilde{F} : \mathbf{R}^m \to \mathbf{R}^{n\times n}$ and $g : \mathbf{R}^m \to \mathbf{R}^{n\times n}$ are affine, namely,
$\tilde{F}(\sum_{i=1}^m \omega_i(t+1)\mathbf{e_i}) = A + \sum_{i=1}^m N_i\omega_i(t+1)$,
$\tilde{g}((\sum_{i=1}^m \omega_i(t+1)\mathbf{e_i}) = \mathbf{a} + \sum_{i=1}^m \mathbf{b}_i\omega_i(t+1)$.
Then we can obtain an Affine Dynamical System $\sigma = ((R^n, F), g, h, h^0)$.

Where F and g are given by the following relations:
$F(0) = A, F(\mathbf{e_i}) = A + N_i(1 \leq i \leq m)$,
$g(0) = \mathbf{a}, g(\mathbf{e_i}) = \mathbf{a} + b_i(1 \leq i \leq m)$.
And U is given by $U = \{\mathbf{0}, \mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_m}\}$ and $e_i = [0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T$.
Where, T denotes the transpose.
Therefore, we can conclude that the assumption for the set U to be finite is not so special.

(5.17) Proposition
Let $\sigma = ((K^n, F), g, h, h^0)$ be Affine Dynamical System. σ is quasi-reachable if and only if
rank $[(g(u_1), g(u_2), \cdots, g(u_m), F(u_1)g(u_1), \cdots, F(u_1)g(u_m), \cdots,$
$F^{n-1}(u_m)g(u_1), \cdots, F^{n-1}(u_m)g(u_m)] = n$.

[proof] See Proposition (5-B.6).

(5.18) Proposition
Let $\sigma = ((K^n, F), g, h, h^0)$ be an Affine Dynamical System. σ is distinguishable
if and only if
rank $[h^T, (hF(u_1)^T, (hF(u_2)^T, \cdots, (hF(u_m))^T, \cdots,$
$(hF^2(u_1))^T, \cdots, (hF^2(u_m))^T, (hF^{n-1}(u_1)g(u_m))^T, \cdots,$
$(hF^{n-1}(u_m)g(u_m))^T] = n$.

[proof] See Proposition (5-B.13).

(5.19) Definition
Let the input value's set U be $U := \{u_i; 1 \leq i \leq m\}$ and let a map $\| \ \| : U \to N$ be $u_i \mapsto \|u_i\| = i$. And let a numerical value $|||\omega|||$ of an input $\omega \in \Omega$ be $|||\omega||| = \|\omega(|\omega|)\| + \|\omega(|\omega| - 1)\| \times m + \cdots + \|\omega(1)\| \times m^{|\omega|-1}$ and $|||\mathit{1}||| = 0$.
Then we can define a totally ordered relation by this numerical value in Ω. Namely, $\omega_1 \leq \omega_2 \Longleftrightarrow |||\omega_1||| \leq |||\omega_2|||$.

(5.20) Definition
Let $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ be a canonical Affine Dynamical System. If input sequences $\{\omega_i \in \Omega; 1 \leq i \leq n\}$ satisfy the following conditions, then σ_s is said to be a quasi-reachable standard system.
1) $\mathbf{e_i} = \sum_{j=1}^{i} F_s(\omega_j(|\omega_j|)F_s(\omega_j(|\omega_j|-1)F_s(\omega_j(|\omega_j|-j)g_s(\omega_j(j))$
2) $1 = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ hold.
3) $\sum_{j=1}^{|\omega|} F_s(\omega(|\omega|)F_s(\omega(|\omega|-1)F_s(\omega(|\omega|-j)g_s(\omega(j)) = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}, \alpha_i \in K$ holds for any input sequence $\omega \in \Omega$ such that $\omega_j < \omega < \omega_{j+1}(1 \leq i \leq n-1)$.

(5.21) Theorem
For any canonical Affine Dynamical System $\sigma = ((K^n, F), g, h, h^0)$, there exists uniquely the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ which is isomorphic to it.

[proof] See Proposition (5-B.16).

(5.22) Definition
Let Y be the field K for convenience. A canonical Affine Dynamical System $\sigma_d = ((K^n, F_d), g_d, \mathbf{e_1}^T, h^0)$ is said to be a distinguishable standard system if input sequences $\{\omega_i \in \Omega; 1 \leq i \leq n\}$ given by $\mathbf{e_i}^T = \mathbf{e_1}^T G_{F_d}(\omega_i)$ satisfy the following conditions:
Where $\mathbf{e_1}^T G_{F_d}(\omega_i) = \sum_{j=1}^{|\omega_i|} F_d(\omega_i(|\omega_i|))F_d(\omega_i(|\omega_i|-1)) \cdots F_d(\omega_i(|\omega_i|-j))$ $\times \cdots F_d(\omega_i(1))$.
1) $1 = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i-1$ for $i(1 \leq i \leq n)$ hold.
2) $\mathbf{e_1}^T G_{F_d}(\omega) = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}^T, \alpha_i \in K$ holds for any input sequence $\omega \in \Omega$ such that $\omega_j < \omega < \omega_{j+1}(1 \leq i \leq n-1)$.

(5.23) Representation Theorem for equivalence classes
For any finite dimensional canonical Linear Representation System, there exists a uniquely determined isomorphic distinguishable standard system.

[proof] See Proposition (5-B.17).

(5.24) Definition
For any input response map $a \in F(U^*, Y)$, we can consider the following infinite matrix H_a^A. The H_a^A is called a Hankel Matrix of a.

$$H_a^A = \bar{\omega}\begin{pmatrix} & & \omega & \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \\ \cdots & \cdots & a(\bar{\omega}\,|\omega) - a(\bar{\omega}) & \\ & & & \end{pmatrix}$$

(5.25) Theorem for existence criterion
For an input response map $a \in F(U^*, Y)$, the following conditions are equivalent:
1) a is a behavior of an n-dimensional canonical Affine Dynamical System.
2) $\{S_l(\omega)a - a : \omega \in U^*\}$ have n linearly independent vectors.
3) rank of H_a^A is n.
Where $S_l(\omega)a - a \in F(U^*, Y)$ is defined by $S_l(\omega)a - a : U^* \to Y; \bar{\omega} \mapsto a(\bar{\omega}|\omega) - a(\bar{\omega})$.

[proof] See Proposition (5-B.18).

(5.26) Theorem for a realization procedure
Let an input response map $a \in F(U^*, Y)$ satisfy the condition of Theorem (5.25), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ which realizes it can be obtained by the following procedure:
1) Select n linearly independent vectors $\{S_l(\omega_i)a - a : (1 \leq i \leq n)\}$ from $\{S_l(\omega)a - a : \omega \in U^*, |\omega| \leq n-1\}$ in the order of the numerical value of U^*.
2) Let the state space be K^n. For the set $\{\omega_j : |\omega_j| = 1\}$ of input sequence, set $g_s(\omega_j) = \mathbf{e_j}$. Moreover, let $g_s(\omega_j) = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}$ for any $\omega \in U^*$ such that $\omega_j < \omega < \omega_{j+1}$ and $|\omega_j| = |\omega_{j+1}| = 1$.
3) Let $h_s = [a(\omega_1) - a(1), a(\omega_2) - a(1), \cdots, a(\omega_n) - a(1)]$.
4) For any $i(1 \leq i \leq n)$, let ${}_if_j$ in $F_s(u_i) = [{}_if_1, {}_if_2, \cdots, {}_if_n] \in K^{n\times n}$ be ${}_if_j = [{}_if_{j1}, {}_if_{j2}, \cdots, {}_if_{jn}]^T$.
Where $S_l(u_i)(S_l(\omega_j)a - a) = \sum_{k=1}^{n} {}_if_{jk}\ (S_l(\omega_k)a - a)$ holds for any $j(1 \leq j \leq n)$.
5) Set $h^0 = a(1)$.

[proof] See Proposition (5-B.19).

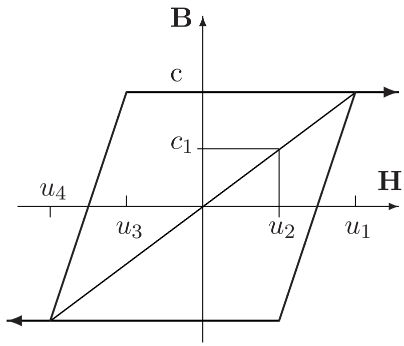


Fig. 5.1. A typical hysteresis characteristic (back-lash) of magnetic field **H** and magnetic flux density **B**

(5.27) Example

As an example of the realization procedure, we will treat the following typical hysteresis characteristic. This is a different one from one treated in Hasegawa and Matsuo [1996a]. Let the magnetic field **H** be input and the magnetic flux density **B** be output.

Applying the above procedure (5.26) as follows, we obtain the following the quasi-reachable standard system $\sigma = ((\mathbf{R}^3, F), g, h, h^0)$ which realizes the given figure.

1) We select the linearly independent vectors $\{S_l(u_1)a - a =: a_1, S_l(u_2)a - a =: a_2, S_l(-u_1)a - a =: a_3\}$.

2) Let the state space be $\mathbf{R}^3$. For the set $\{u_1, u_2, -u_1\}$ of input sequence, set $g_s(u_1) = \mathbf{e_1}, g_s(u_2) = \mathbf{e_2}, g_s(-u_1) = \mathbf{e_3}$. And set $g_s(-u_2) = -\mathbf{e_2}$

And we set following:

$g_s(p) = \frac{p-u_2}{u_1-u_2}\mathbf{e}_1 + \frac{u_1-p}{u_1-u_2}\mathbf{e}_2$ for p in $u_2 \leqq p \leqq u_1$,

$g_s(q) = \frac{q}{u_2}\mathbf{e}_2$ for q in $u_3 \leqq q \leqq u_2$,

$g_s(r) = \frac{u_4-r}{u_3-u_4}\mathbf{e}_2 + \frac{u_3-r}{u_3-u_4}\mathbf{e}_3$ for r in $u_4 \leqq r \leqq u_3$

Because the following hold.

$\mathrm{S}_l(-u_2)a - a = -a_2$.

$\mathrm{S}_l(p)a - a = ((p-u_2)/(u_1-u_2))a_1 + ((u_1-p)/(u_1-u_2))a_2$ for p in $u_2 \leqq p \leqq u_1$.

$\mathrm{S}_l(q)a - a = (q/u_2)a_2$ for q in $u_3 \leqq q \leqq u_2$.
$\mathrm{S}_l(r)a - a = ((u_4 - r)/(u_3 - u_4))a_2 + ((u_3 - r)/(u_3 - u_4))a_3$ for r in $u_4 \leqq r \leqq u_3$.
3) Set $h_s \in \mathbf{R}^{1\times 3}$ be $h_s = [c, c_1, -c]$
4) Let $F_s(p), F_s(q_1), F_s(q_2), F_s(r) \in \mathbf{R}^{3\times 3}$ be the following:
Where p $(u_2 \leqq p \leqq u_1)$, q_1 $(-u_1 * \frac{c_1^2}{c^2} \leqq q_1 \leqq u_2)$, q_2 $(u_3 \leqq q_2 \leqq -u_1 * \frac{c_1^2}{c^2})$, r $(u_4 \leqq r \leqq u_3)$.

$$F_s(p) = \begin{bmatrix} \frac{u_1-p}{u_1-u_2} & 0 & 0 \\ \frac{u_1-p}{u_1-u_2} & 0 & \frac{p-u_1}{u_1-u_2} \\ 0 & 0 & \frac{u_1-p}{u_1-u_2} \end{bmatrix}, \; F_s(q_1) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-q_1}{u_2} & \frac{u_2-q_1}{u_2} & \frac{-q_1}{u_2} \\ 0 & 0 & 1 \end{bmatrix},$$
$$F_s(q_2) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-q_2}{u_2} & \frac{(c+c_1)(q_2+u_2)}{c_1(u_1-u_2)} & \frac{-q_2}{u_2} \\ 0 & 0 & 1 \end{bmatrix}, \; F_s(r) = \begin{bmatrix} \frac{u_1+r}{u_1-u_2} & 0 & 0 \\ \frac{u_1+r}{u_1-u_2} & 0 & \frac{u_4-r}{u_3-u_4} \\ 0 & 0 & \frac{r-u_4}{u_3-u_4} \end{bmatrix}.$$

Especially, $F_s(u_i) \in \mathbf{R}^{3\times 3}$ for i in $(1 \leq i \leq 4)$ are given by $F_s(u_1) = \mathbf{0}, F_s(u_4) = \mathbf{0}$, and $F_s(u_2)$, $F_s(u_3)$ are given by the following:
$$F_s(u_2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \; F_s(u_3) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the following hold:
$\mathrm{S}_l(p)(\mathrm{S}_l(u_1)a - a) = \frac{u_1-p}{u_1-u_2}a_1 + \frac{u_1-p}{u_1-u_2}a_2$,
$\mathrm{S}_l(p)(\mathrm{S}_l(u_2)a - a) = \mathbf{0}$,
$\mathrm{S}_l(p)(\mathrm{S}_l(u_4)a - a) = \frac{p-u_1}{u_1-u_2}a_2 + \frac{u_1-p}{u_1-u_2}a_3$,
$\mathrm{S}_l(q_1)(\mathrm{S}_l(u_1)a - a) = a_1 - \frac{q_1}{u_2}a_2$,
$\mathrm{S}_l(q_1)(\mathrm{S}_l(u_2)a - a) = \frac{u_2-q_1}{u_2}a_2$,
$\mathrm{S}_l(q_1)(\mathrm{S}_l(u_4)a - a) = \frac{-q_1}{u_2}a_2 + a_3$,
$\mathrm{S}_l(q_2)(\mathrm{S}_l(u_1)a - a) = a_1 - \frac{q_2}{u_2}a_2$,
$\mathrm{S}_l(q_2)(\mathrm{S}_l(u_2)a - a) = \frac{(c+c_1)(q_2+u_2)}{c_1(u_1-u_2)}a_2$,
$\mathrm{S}_l(q_2)(\mathrm{S}_l(u_4)a - a) = \frac{-q_2}{u_2}a_2 + a_3$,
$\mathrm{S}_l(r)(\mathrm{S}_l(u_1)a - a) = \frac{u_1+r}{u_1-u_2}a_1 + \frac{u_1+r}{u_1-u_2}a_2$,
$\mathrm{S}_l(r)(\mathrm{S}_l(u_2)a - a) = \mathbf{0}$,
$\mathrm{S}_l(r)(\mathrm{S}_l(u_4)a - a) = \frac{u_4-r}{u_3-u_4}a_2 + \frac{r-u_4}{u_3-u_4}a_3$
5) $h^0 = a(1) = 0$

Remark: An expression of a typical hysteresis characterestic by a difference equation is more suitable for a computer algorithm than by a usual method of a describing function.

5.3 Historical Notes and Concluding Remarks

We characterized the finite-dimensionality of Affine Dynamical Systems, which realize any input response map (equivalently, input/output map with causality). Also we obtained the following same results as in linear system theory.

A criterion for being canonical of finite-dimensional Affine Dynamical Systems was given. There uniquely exists the quasi-reachable standard system in the isomorphic class of finite-dimensional canonical Affine Dynamical Systems. We obtained a criterion for the behavior of finite-dimensional Affine Dynamical Systems. We also gave a procedure to obtain the quasi-reachable standard system from the input response map.

In the reference [Tarn and Nonoyama 1979], an existence condition of biaffine system is given by the generalized Hankel Matrix. Since an insufficient morphism between biaffine system and homogeneous bilinear systems is introduced, there is a wrong condition for the existence theorem in the reference. Therefore, there is a wrong proof in the uniqueness theorem. Note that the Hankel Matrix of this paper is different from the Hankel Matrix of the reference, in spite of the clear relation between our Affine Dynamical Systems and biaffine systems [See (5.15) biaffine systems]. Note that the following conditions hold:
rank H_a^A =rank H_a^L or rank H_a^A +1 =rank H_a^L hold. Where H_a^A is the infinite Hankel matrix for Affine Dynamical Systems, H_a^L is the infinite Hankel matrix for Linear Representation Systems discussed in Chapter 4.
Note that homogeneous bilinear systems are a subclass of our Linear Representation Systems.

We will consider the following dynamical system:

$$\begin{cases} x(t+1) &= (A+\sum_{i=1}^{m}\mathbf{N}_i\cdot\omega_i(t+1))x(t)+\sum_{i=1}^{m}\bar{g}\cdot\omega_i(t+1) \\ x(0) &= 0 \\ \gamma(t) &= h^0+hx(t) \end{cases}$$

$\omega_i(t)\in\mathbf{R}$, x(t), $\bar{g}\in\mathbf{R}^n$, $\mathbf{N}_i\in\mathbf{R}^{n\times n}$ and $\gamma(t)\in Y$.

Let $F(\omega(t+1)) = A + \sum_{i=1}^{m} \omega_i(t+1)\mathbf{N}_i$, $g(\omega(t+1)) = \sum_{i=1}^{m} \bar{g}\omega_i(t+1)$, then the above dynamical system is an Affine Dynamical System (see also (4.9) in Chapter 4). Therefore, the inhomogeneous bilinear system is an example of our Affine Dynamical Systems. See D'alessandro, Isidori and Ruberti [1974] for ones of continuous-time. See Isidori [1973] and Tarn and Nonoyama [1979] for ones of discrete-time. D'alessandro, Isidori and Ruberti [1974] only gave a criterion for existence of finite dimensional inhomogeneous bilinear systems in continuous-time.

Isidori [1973] gave a sufficient condition for the existence and an algorithm for the partial realization problem. Tarn and Nonoyama [1976] gave a wrong existence condition and a wrong uniqueness condition because of misunderstanding system morphisms in inhomogeneous bilinear systems. For this, see Niinomi and Matsuo [1981].

As we mentioned in 4.4 of Chapter 4, we have two dynamical systems which realize arbitrary input response map (equivalently, any input/output map with causality), which are Linear Representation Systems and Affine Dynamical Systems. Details of relations between the two dynamical systems were discussed in Niinomi and Matsuo [1981]. Sontag [1979a] presented realization theorem of state-affine systems which are a little different from our Affine Dynamical Systems. However, he did not give an initial object of the category. Hence, he could not introduce Hankel matrix.

5.4 Appendix

5.4.A Realization Theorem

5.4.A.1 Fundamentals for Affine Dynamical Systems

(5-A.1) Definition
Let X be a set and $\mathbf{X}$ be a linear space over the field K. If the following conditions hold, then a pair $(X, \mathbf{X})$ is called an affine space. The pair $(X, \mathbf{X})$ may be written as $X^A = (X, \mathbf{X})$.
1) For any $x_1, x_2 \in X$, there exists uniquely one $\mathbf{x} \in \mathbf{X}$ such that $x_2 = \mathbf{x} + x_1$.
2) For any $\mathbf{x} \in \mathbf{X}$ and $x_1, x_2 \in X$, $x_2 + (x_1 + \mathbf{x}) = (x_2 + x_1) + \mathbf{x}$ holds.

Remark: Let $X^A = (X, \mathbf{X})$ be an affine space. For any a point $x \in X$ and a fixed element x^0, there exists $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x}$ can be uniquely expressed by $x = \mathbf{x} + x^0$. Therefore, a collection (X^A, x^0) is said to be a pointed affine space.

(5-A.2) Definition
Let $X_1^A = (X_1, \mathbf{X_1})$ and $X_2^A = (X_2, \mathbf{X_2})$ be affine spaces. If a map $f : X_1 \to X_2$ and a map $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ satisfy the following conditions, then a pair $(f, \mathbf{f})$ is called an affine map $(f, \mathbf{f}) : X_1^A \to X_2^A$. The pair $(f, \mathbf{f})$ may be written by $f^A := (f, \mathbf{f})$.
1) A map $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ is a linear map.
2) For any $x \in X_1$ and $\mathbf{x} \in \mathbf{X_1}$, the equation $f(x + \mathbf{x}) = f(x) + \mathbf{f}\mathbf{x}$ holds.
This $\mathbf{f}$ is called a linear map induced by f.

(5-A.3) Definition
Let $(X_1, \mathbf{X_1}, x_1^0)$ and $(X_2, \mathbf{X_2}, x_2^0)$ be pointed affine spaces. If an affine map $(f, \mathbf{f}) : (X_1, \mathbf{X_1}) \to (X_2, \mathbf{X_2})$ satisfies $f(x_1^0) = x_2^0$, then $(f, \mathbf{f})$ is said to be a pointed affine map : $(X_1, \mathbf{X_1}, x_1^0) \to (X_2, \mathbf{X_2}, x_2^0)$.

(5-A.4) Proposition
Let $(X_1, \mathbf{X_1}, x_1^0)$ and $(X_2, \mathbf{X_2}, x_2^0)$ be pointed affine spaces. And let $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ be a linear map. An affine map $(f, \mathbf{f}) : (X_1, \mathbf{X_1}, x_1^0) \to (X_2, \mathbf{X_2}, x_2^0)$ is a pointed affine map if and only if $f(\mathbf{x} + x_1^0) = \mathbf{f}\mathbf{x} + x_2^0$ holds for any $\mathbf{x} \in \mathbf{X_1}$.

[proof] This is obtained by the definition of the affine map and the pointed affine map.

(5-A.5) Definition
Let $(f, \mathbf{f}) : (X_1, \mathbf{X_1}) \to (X_2, \mathbf{X_2})$ be an affine map. If f is surjective, then the affine map $(f, \mathbf{f})$ is called epimorphism.
If f is injective, then the affine map $(f, \mathbf{f})$ is called monomorphism.
If f is bijective, then the affine map $(f, \mathbf{f})$ is called isomorphism.

(5-A.6) Proposition
The following facts about affine maps hold:
1) An affine map $(f, \mathbf{f}) : (X_1, \mathbf{X_1} \to (X_2, \mathbf{X_2})$ is epimorphism if and only if the linear map $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ is surjective.
2) An affine map $(f, \mathbf{f}) : (X_1, \mathbf{X_1} \to (X_2, \mathbf{X_2})$ is monomorphism if and only if the linear map $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ is injective.
3) An affine map $(f, \mathbf{f}) : (X_1, \mathbf{X_1} \to (X_2, \mathbf{X_2})$ is isomorphism if and only if the linear map $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ is bijective.

[proof] These are obtained by the definition of affine spaces and affine maps.

(5-A.7) Definition
Let $(X, \mathbf{X})$ be an affine space. And let Y be a sub set of X, and $\mathbf{Y}$ be a sub space of $\mathbf{X}$. If the following condition folds, then a pair $(Y, \mathbf{Y})$ is called a sub affine space of $(X, \mathbf{X})$.
Condition: For any $y \in Y$ and $\mathbf{y} \in \mathbf{Y}$, $y + \mathbf{y} \in Y$ holds.

(5-A.8) Definition
Let $(X, \mathbf{X})$ be an affine space. For any sub set $X_h \subset X$, the smallest affine space that contains X_h among sub affine spaces $(Y, \mathbf{Y})$ such that $Y \supset X_h$ is called an affine hull of X_h.

5.4.A.2 Affine State Structure: Linear U-Actions

In Appendix 3.5.B of Chapter 3 we have introduced the pointed U-action and U-actions with an input map, and we have shown that they are equivalent. In this section we introduce pointed linear U-actions and linear U-action with a linear input map, and show that they are equivalent. Moreover, we discuss reachability of pointed linear U-actions.

(5-A.9) Definition

A system given by the following equation is written as a pair (X^A, F^A) and it is said to be an affine U-action.

$x(t+1) = F^A(\omega(t+1)x(t)$.

Where $X^A = (X, \mathbf{X})$ is an affine space over the field K and a map $F^A = (F, \mathbf{F}) : U \to AM(X^A); u \mapsto F^A(u)$. $AM(X^A) := \{\text{any affine map} : X^A \to X^A\}$

Let (X_1^A, F_1^A) and (X_2^A, F_2^A) be affine U-actions, then an affine map $T^A : X_1^A \to X_2^A$ is said to be an affine U-morphism $(X_1^A, F_1^A) \to (X_2^A, F_2^A)$, if T^A satisfies $T^A F_1^A(u) = F_2^A(u) T^A$ for any $u \in U$.

For an affine U-action (X^A, F^A) and a state $x^0 \in X$, a collection $((X^A, F^A), x^0)$ is said to be a pointed affine U-action. It represents the following equation:

$$\begin{cases} x(t+1) & = & F^A(\omega(t+1))x(t) \\ x(0) & = & x^0 \end{cases}$$

For a pointed affine U-action $((X^A, F^A), x^0)$, if the affine hull of $\{F^A(\omega(|\omega|)) \times F^A(\omega(|\omega|-1)) \cdots F^A(\omega(1))x^0; \omega \in \Omega\}$ is equal to X^A, then $((X^A, F^A), x^0)$ is said to be quasi-reachable.

Let $((X_1^A, F_1^A), x_1^0)$ and $((X_2^A, F_2^A), x_2^0)$ be pointed affine U-actions, then an affine U-morphism $T^A = (T, \mathbf{T}) : ((X_1^A, F_1^A), x_1^0) \to ((X_2^A, F_2^A), x_2^0)$ which satisfies $T(x_1^0) = x_2^0$ is said to be a pointed affine U-morphism : $((X_1^A, F_1^A), x_1^0) \to ((X_2^A, F_2^A), x_2^0)$.

Remark: Proposition (5.2) implies that a pointed affine U-action $((X^A, F^A), x^0)$ may be written by the following equation:

$$\begin{cases} x(0) & = & x^0 \\ x(t+1) & = & \mathbf{x(t+1)} + x^0 \\ \mathbf{x(t+1)} & = & F(\omega(t+1))x(t) + g(\omega(t+1)) \end{cases}$$

Where $X^A = (X, \mathbf{X}), F^A = (F, \mathbf{F})$. And $x^0 \in X, x(t) \in X, \mathbf{x(t)} \in \mathbf{X}$ and a map $g : U \to \mathbf{X}$ for any $t \in N$.

(5-A.10) Definition

A collection $((\mathbf{X}, \mathbf{F}), g)$ given by the following equations is said to be a linear U-action with an affine map.

$$\begin{cases} \mathbf{x(0)} & = & \mathbf{0} \\ \mathbf{x(t+1)} & = & \mathbf{F}(\omega(t+1))\mathbf{x(t)} + g(\omega(t+1)) \end{cases}$$

Where $\mathbf{x(t)} \in \mathbf{X}$, $\mathbf{F}(\omega(t)) \in L(\mathbf{X})$, $\omega(t) \in U$ and $t \in N$.

(5-A.11) Lemma
Any pointed affine U-action $((X^A, F^A), x^0)$ is isomorphic to a linear U-action with an affine map $((\mathbf{X}, \mathbf{F}), g)$.

[proof] Let's consider a pointed affine U-morphism $T^A = (T, \mathbf{I}) : X^A = (X, \mathbf{X}) \rightarrow (X, \mathbf{X})$ such that $T(x^0) = 0$ and $\mathbf{I}(\mathbf{x}) = \mathbf{x}$ for any $\mathbf{x} \in \mathbf{X}$. Then $T^A = (T, \mathbf{I})$ is a pointed U-morphism : $((X^A, F^A), x^0) \rightarrow ((\mathbf{X}, \mathbf{F}), g)$, and it is an isomorphic map : $X^A = (X, \mathbf{X}) \rightarrow (X, \mathbf{X})$.

Remark: By following this Lemma (5-A.11), from now on we may use $((X, F), g)$ in no bold fonts in place of using $((\mathbf{X}, \mathbf{F}), g)$ in bold fonts.

(5-A.12) Definition
Let $((X_1, F_1), g_1)$ and $((X_2, F_2), g_2)$ be linear U-actions with an affine map. If a linear U-morphism $T : (X_1, F_1) \rightarrow (X_2, F_2)$ which satisfies $Tg_1 = g_2$, then T is said to be a linear U-morphism with an affine map : $((X_1, F_1), g_1) \rightarrow ((X_2, F_2), g_2)$.

(5-A.13) Example
Let $V(\Omega^+)$ and ψ be the same as that considered in Example (5.12). And we consider a map $e : U \rightarrow V(\Omega^+)$ considered in Example (5.12) is a linear map : $V(\Omega^+) \rightarrow V(\Omega^+)$. Then $((V(\Omega^+), \psi), e)$ is a linear U-action with an affine map.

(5-A.14) Example
Let $F(\Omega, Y)$ be a set of any input response maps $a : \Omega \rightarrow Y$, and S_l be the same as in Example (5.13). Let a map $\xi : U \rightarrow (F(\Omega, Y)$, then $(F(\Omega, Y), S_l)$ is a linear U-action and $((F(\Omega, Y), S_l), \xi)$ is a linear U-action with an affine map.

(5-A.15) Definition
Let $((X, F), g)$ be any linear U-action with an affine map. A linear U-morphism with an affine map $G : ((V(\Omega^+), \psi), e) \rightarrow ((X, F), g)$ is said to be an affine input map.
A linear U-morphism with an affine map:
$A : ((V(\Omega^+), \psi), e) \rightarrow ((F(\Omega, Y), S_l), \xi)$ is said to be an affine input/output map.

(5-A.16) Proposition
For any linear U-action with an affine map $((X, F), g)$, there uniquely exists an affine input map $G : ((V(\Omega^+), \psi), e) \to ((X, F), g)$.
Conversely, for any linear U-morphism $G : (V(\Omega^+), \psi) \to (X, F)$, $((X, F), g)$ is a linear U-action with an affine map. Where g is given by a map $g : U \to X$ which satisfies $g(u) = G(e_u)$.

[proof] First we will prove the first half. An affine map G satisfies the following equation.
$G(e_\omega) = \sum_{j=1}^{|\omega|} F(\omega(|\omega|))F(\omega(|\omega| - 1)) \cdots F(\omega(j + 1)g(\omega(j)).$
Since $\{e_\omega; \omega \in \Omega^+\}$ is a basis in $V(\Omega^+)$, a linear map $G : V(\Omega^+) \to X$ satisfying the above equation is uniquely determined on the linear space $V(\Omega^+)$.
The latter half is easily obtained by Definition (5-A.10).

Remark: In Proposition (5-A.16), if we replace the linear U-action with an affine map $((F(\Omega, Y), S_l), \xi)$ considered in Example (5-A.14) in place of $((X, F), g)$, then note that an affine input map $G : ((V(\Omega^+), \psi), e) \to ((F(\Omega, Y), S_l), \xi)$ is an affine input/output map.

(5-A.17) Definition
For a linear U-action with an affine map $((X, F), g)$ and an affine input map G corresponding to it, a collection $((X, F), G)$ is said to be a linear U-action with an affine input map.

Remark: According to Proposition (5-A.16), a category of a linear U-action with an affine map $((X, F), g)$ is equivalent to a category of a linear U-action with an affine input map $((X, F), G)$.

(5-A.18) Proposition
Let $((X, F), G)$ be a linear U-action with an affine input map corresponding to a linear U-action with an affine map $((X, F), g)$. Then $((X, F), g)$ is quasi-reachable if and only if G in $((X, F), G)$ is surjective.

[proof] By the definition of quasi-reachablity, this can be obtained easily.

5.4.A.3 Affine State Structure with a Readout Map

In Appendix 3.5 of Chapter 3 we have introduced the U-action with a readout map and U-actions with an observation map, we have shown that they are equivalent. In this section we introduce linear U-actions with a readout map and linear U-actions with a linear output map, and show that they are equivalent. Moreover, we discuss distinguishability of linear U-actions with a readout map.

(5-A.19) Definition
A system given by the following equations is written by $((X^A, F^A), h^A)$, and it is said to be an affine U-action with a readout map.

$$\begin{cases} x(t+1) & = & F^A(\omega(t+1))x(t) \\ \gamma(t) & = & h^A x(t) \end{cases}$$

for any $t \in N$, where $x(t) \in X^A$ and $\gamma(t) \in Y^A$.
Where $X^A = (X, \mathbf{X})$ is an affine space that may be called a state space, F^A is an affine map : $U \to AM(X^A); u \mapsto F^A(u)$ and $h^A : X^A \to Y^A$ is an affine map. And $Y^A = (Y, \mathbf{Y})$. See Appendix 5.4.A.1.
If $h^A F^A(\omega(|\omega|) F^A(\omega(|\omega|-1) \cdots F^A(\omega(1)x_1$
$= h^A F^A(\omega(|\omega|) F^A(\omega(|\omega|-1) \cdots F^A(\omega(1)x_2$ implies $x_1 = x_2$,
then $((X^A, F\ A), h^A)$ is called distinguishable.
Let $((X_1^A, F_1^A), h_1^A)$ and $((X_2^A, F_2^A), h_2^A)$ be affine U-actions with readout map. If an affine map $T^A = (T, \mathbf{T}) : X_1^A \to X_2^A$ satisfies $T^A F_1^A(u) = F_2^A(u) T^A$ and $h_1^A = h_2^A T^A$, then T^A is said to be an affine U-morphism with a readout map $T^A : ((X_1^A, F_1^A), h_1^A) \to ((X_2^A, F_2^A), h_2^A)$.

(5-A.20) Proposition
Let $((X_1^A, F_1^A), h_1^A)$ and $((X_2^A, F_2^A), h_2^A)$ be affine U-actions with a readout map. And let $X_i^A := (X_i, \mathbf{X_i})$, $F_i^A = (F_i, \mathbf{F_i})$ and $h_i^A = (h_i, \mathbf{h_i})$ (for $i = 1, 2$), and let $T^A = (T, \mathbf{T})$ be an affine map : $X_1^A \to X_2^A$. Then $T^A = (T, \mathbf{T})$ is an affine U-morphism with a readout map $T^A : ((X_1^A, F_1^A), h_1^A) \to ((X_2^A, F_2^A), h_2^A)$ if and only if $\mathbf{T}\mathbf{F_1}(u) = \mathbf{F_2}(u)\mathbf{T}, \mathbf{h_1} = \mathbf{h_2}\mathbf{T}$ and $\mathbf{T}$ is a linear U-morphism with a readout map $\mathbf{T} : ((\mathbf{X_1}, \mathbf{F_1}), \mathbf{h_1}) \to ((\mathbf{X_2}, \mathbf{F_2}), \mathbf{h_2})$, namely $\mathbf{T}$ is a linear map $\mathbf{X_1} \to \mathbf{X_2}$ such that $\mathbf{T}\mathbf{F_1}(u) = \mathbf{F_2}(u)\mathbf{T}$ and $\mathbf{h_1} = \mathbf{h_2}\mathbf{T}$ for any $u \in U$.

[proof] Direct calculations can show this.

In Appendix 5.4.A.2, we have shown that the state space in an Affine Dynamical System can be condensed to the linear space. Also, by Proposition (5-A.20), we will remark the linear parts in Affine Dynamical Systems. Hence, we can rewrite Definition (5-A.19) for an affine U-action with output as the following:

(5-A.21) Definition
For a linear U-action $(\mathbf{X}, \mathbf{F})$ and a linear map $\mathbf{h} : \mathbf{X} \to Y$, a collection $((\mathbf{X}, \mathbf{F}), \mathbf{h})$ is said to be a linear U-action with a readout map.
If $\mathbf{hF}(\omega(|\omega|))\mathbf{F}(\omega(|\omega|-1))\cdots\mathbf{F}(\omega(1))x_1 = \mathbf{hF}(\omega(|\omega|))\mathbf{F}(\omega(|\omega|-1))\cdots \times \mathbf{F}(\omega(1))x_2$ for any $\omega \in \Omega$ implies $x_1 = x_2$, then a linear U-action with a readout map $((\mathbf{X}, \mathbf{F}), \mathbf{h})$ is said to be distinguishable.

Remark: By following Proposition (5-A.20) and Definition (5-A.21), from now on we may use a linear U-action with a readout map $((X, F), h)$ in no bold fonts in place of using $((\mathbf{X}, \mathbf{F}), \mathbf{h})$ in bold fonts.

(5-A.22) Example
Let $F(\Omega, Y)$ be a set of any input response maps and S_l be defined in Example (5.13). Then $(F(\Omega, Y), S_l)$ is a linear U-action. Let 1 be a linear operator $: F(\Omega, Y) \to Y; a \mapsto a(\mathit{1})$. Then a collection $((F(\Omega, Y), S_l), 1)$ is a linear U-action with a readout map.

We have introduced linear U-actions with a readout map and a linear observation map in Definition (4-A.32) of Chapter 4. Here we state it clearly as the following definition:

(5-A.23) Definition
Let (X, F) be a linear U-action and H be a linear U-morphism : $(X, F) \to (F(\Omega, Y), S_l)$, then a collection $((X, F), H)$ is said to be a linear U-action with an observation map.

(5-A.24) Proposition
For any linear U-action with a readout map $((X, F), h)$, there uniquely exists a linear U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$ such that $Hx(\omega) = hF(\omega(|\omega|)F(\omega(|\omega|-1)\cdots F(\omega(1))x$ for any $x \in X$ and $\omega \in \Omega$.
Conversely, for any linear U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$, an operator h: $X \to Y$ that is defined by $h = 1 \cdot H$ is a linear operator.

[proof] This proposition is the same as Proposition (4-A.32) in Appendix 4.5.

Remark: Proposition (5-A.24) implies that a category of any linear U-action with a readout map is isomorphic to a category of any linear U-action with an observation map.

(5-A.25) Proposition
Let $((X,F),H)$ be a linear U-action with an observation map corresponding to a linear U-action with a readout map $((X,F),h)$. Then $((X,F),h)$ is distinguishable if and only if H in $((X,F),H)$ is injective.

[proof] By the definition of distinguishability, this can be obtained easily.

(5-A.26) Lemma
Let $F_0(\Omega,Y) := \{a \in F(\Omega,Y); a(1) = 0\}$. Then $F_0(\Omega,Y) = L(V(\Omega^+),Y)$.

[proof] We will consider a map $:F_0(\Omega,Y) \to L(V(\Omega^+),Y); a \mapsto a_l$, where $a(\omega) = a_l(e_\omega)$ holds for any $\omega \in \Omega^+$. Then the map is clearly bijective.

(5-A.27) Example
Let $V(\Omega^+)$ be a set and ψ be the operator in Example (5.12). Let $a_l \in L(V(\Omega^+),Y)$ introduced in Lemma (5-A.26) and its proof. Then a collection $((V(\Omega^+),\psi),a_l)$ is a linear U-action with a readout map.

Remark: If we replace $((V(\Omega^+),\psi),a_l)$ with $((X,F),h)$ in Proposition (5-A.24), then note that a linear U-morphism $A : (V(\Omega^+),\psi) \to (F(\Omega,Y),S_l)$ is an affine input/output map.

As the structure of Linear Representation Systems, we have introduced linear U-actions and linear Ω-modules in the Appendix 4.5 of Chapter 4. Also we have clarified the relation between linear U-actions and linear Ω-modules. In this chapter we were also able to introduce linear U-actions as the structure of Affine Dynamical Systems. Therefore, the relation between linear U-action and Ω-module in Affine Dynamical Systems are the same as before. See Appendix 4.5 in Chapter 4 for it.

In Appendix 4.5 of Chapter 4 we have introduced the sub linear U-action, the quotient linear U-action and the product linear U-action. For Affine Dynamical Systems, we can also consider them. Next, we will list them again for readability.

(5-A.28) Sub linear U-actions
Let (X, F) be a linear U-action and $Y \subseteq X$ be invariant sub-space under F, i.e. $F(u)y \in Y$ for any $u \in U$ and any $y \in Y$. Let $F_Y(u) := F(u)|_Y$ (restriction of the map $F(u)$ to Y), then (Y, F_Y) is a linear U-action, and it is said to be a sub linear U-action of (X, F).

(5-A.29) Quotient linear U-action
Let (X, F) be a linear U-action and a linear equivalence relation R in X be consistent with F. Namely, an equivalence relation R is given by $x_1 \; R \; x_2 \iff x_1 - x_2 \in S$ for some linear sub space $S \subseteq X$, and $x_1 \; R \; x_2$ implies $F(u)x_1 \; R \; F(u)x_2$ for any $u \in U$. Then we can consider a quotient linear space $X/_R = X/S$. Therefore, we can obtain a quotient linear U-action $(X/Z, \tilde{F})$. Where $\tilde{F}(u) : X/Z \to X/Z; [x] \mapsto [F(u)x]$ for any $u \in U$.

(5-A.30) Corollary
Any linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ can be normally decomposed into $X_1 \xrightarrow{\pi} X_1/\ker T \xrightarrow{T^b} \mathrm{im}\, T \xrightarrow{j} X_2$, where π is the canonical surjection, T^b is the isomorphism associated with T, j is the canonical injection and they are linear U-morphisms respectively.

(5-A.31) Product linear U-actions
Let (X_1, F_1) and (X_2, F_2) be linear U-actions and define $(F_1 \times F_2)(u) : X_1 \times X_2 \to X_1 \times X_2; (x_1, x_2) \mapsto (F_1(u)x_1, F_2(u)x_2)$ for the product space $X_1 \times X_2$ and any $u \in U$. Therefore, $(X_1 \times X_2, F_1 \times F_2)$ is a linear U-action, and it is said to be a product linear U-action of (X_1, F_1) and (X_2, F_2).

5.4.A.4 Affine Dynamical Systems

In this section we introduce sophisticated Affine Dynamical Systems, and show that Affine Dynamical Systems (said to be a naive Affine Dynamical Systems) introduced in Definition (5.1) and sophisticated Affine Dynamical Systems can be considered as the same thing.

(5-A.32) Definition
A collection $\Sigma = ((X, F), G, H, h^0)$ is said to be a sophisticated Affine Dynamical System, if G is a linear U-morphism $G : (V(\Omega^+), \psi) \to (X, F)$ and H is a linear U-morphism : $(X, F) \to (F(\Omega, Y), S_l)$.
For a linear input/output map $A_\Sigma = H \cdot G : (V(\Omega^+), \psi) \to (F(\Omega, Y), S_l)$, (A_Σ, h^0) is said to be the behavior of Σ. For a linear input/output map A

and some $a(1) \in Y$, if $A_\Sigma = A$ and $h^0 = a(1)$, then sophisticated Affine Dynamical System Σ is called a realization of $(A, a(1))$.
A sophisticated Affine Dynamical System $\Sigma = ((X, F), G, H, h^0)$ is called canonical if G is surjective and H is injective.

For $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$, a linear U-morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $TG_1 = G_2$, $H_1 = H_2T$ is said to be a sophisticated Affine Dynamical System morphism : $\Sigma_1 \to \Sigma_2$. If T is surjective and injective then $T : \Sigma_1 \to \Sigma_2$ is said to be an isomorphism.

(5-A.33) Example

For the linear U-action $(V(\Omega^+), \psi)$ in Example (5.12), identity map I on $V(\Omega^+)$ and a linear input/output map $A : (V(\Omega+), \psi) \to (F(\Omega, Y), S_l)$, a collection $((V(\Omega^+), \psi), I, A, a(1))$ is a sophisticated Affine Dynamical System that realizes $(A, a(1))$.
For the linear U-action $(F(\Omega, Y), S_l)$ in Example (5.13), a linear input/output map A and identity map I on $F(\Omega, Y)$, then a collection $((F(\Omega, Y), S_l), A, I, a(1))$ is a sophisticated Affine Dynamical System that realizes $(A, a(1))$.

In this situation, we consider the relation between sophisticated Affine Dynamical Systems and naive ones.

(5-A.34) Proposition

For any sophisticated Affine Dynamical System $\Sigma = ((X, F), G, H, h^0)$, there uniquely exists a naive Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ corresponding to the sophisticated Affine Dynamical System Σ by two equations (a.1) and (a.2) for any $x \in X$ and $\omega \in \Omega$.

$G(e_\omega) = \sum_{j=1}^{|\omega|} F(\omega(|\omega|)F(\omega(|\omega| - 1) \cdots F(\omega(j+1))g(\omega(j))$ .... (a.1)

$Hx(\omega) = hF(\omega(|\omega|)F(\omega(|\omega| - 1) \cdots F(\omega(1))x$ .... (a.2)

This correspondence is isomorphic in the category's sense.

[proof] It is easily obtained from Proposition (5-A.16), Remark in Definition (5-A.17) and Remark in Proposition (5-A.24).

5.4.A.5 Sophisticated Realization Theorem

In this section we will prove Realization Theorem (5.15) for (naive) Affine Dynamical Systems. According to the Remark in Proposition (5-A.16) (or the Remark in Example (5-A.27)) and Proposition (5-A.34), the realization theorem can be replaced with following Theorem (5-A.35). Therefore, proving this theorem implies proving Realization Theorem (5.15).

(5-A.35) (Sophisticated) Realization Theorem
For any affine input/output map $A : (V(\Omega^+), \psi) \to (F(\Omega, Y), S_l)$, there exist at least two sophisticated canonical Affine Dynamical Systems that realize $(A, a(1)$ (existence part).
Let $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$ be sophisticated canonical Affine Dynamical Systems that have same behavior, then there exists an isomorphism $T : \Sigma_1 \to \Sigma_2$ (uniqueness part).

[proof] A next Corollary (5-A.36) signifies proving the existence part. Moreover, Remark in Lemma (5-A.40) signifies proving the uniqueness.

(5-A.36) Corollary
For any affine input/output map $A : (V(\Omega+), \psi) \to (F(\Omega, Y), S_l)$ and $a(1) \in Y$, the following sophisticated Affine Dynamical Systems (1) and (2) are canonical realizations of $(A, a(1))$.
(1) $\Sigma_q = ((V(\Omega^+)/\text{ker } A, \tilde{\psi}), \pi, A^i, a(1))$.
Where π is the canonical surjection : $V(\Omega^+) \to V(\Omega^+)/\text{ker } A$ and A^i is given by $A^i = j \cdot A^b$ for $A^b : V(\Omega^+)/\text{ker } A \to \text{im } A$ being isomorphic with A and j being the canonical injection : $\text{im } A \to F(\Omega, Y)$.
(2) $\Sigma_s = ((\text{im } A, S_l), A^s, j, a(1))$.
Where $A^s = A^b \cdot j$.

[proof] This can be obtained easily by Corollary (5-A.30), Example (5-A.33), the definition of canonicality and behavior.

Next, to prove the uniqueness part of Theorem (5-A.35), we introduce the following morphism $Mor(\Sigma_1, \Sigma_2)$ from a sophisticated Affine Dynamical System Σ_1 to another sophisticated Affine Dynamical System Σ_2. Where $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$ are the same as in Matsuo [1981].
$Mor(\Sigma_1, \Sigma_2) := \{\text{a relation } T : X_1 \to X_2; GrT_{12}^{min} \subseteq GrT_{12} \subseteq GrT_{12}^{max}\}$.

Where GrT_{12}^{min}, GrT_{12} and GrT_{12}^{max} denote the graph of $T_{12}^{min} := G_2 \cdot G_1^{-1}$, T_{12} and $H_2^{-1} \cdot H_1$ respectively. Why this morphism is introduced depends on the next lemma.

(5-A.37) Lemma
$A_{\Sigma_1} = A_{\Sigma_2}$ if and only if $Mor(\Sigma_1, \Sigma_2) \neq \emptyset$.

[proof] This can be proved the same as in Matsuo [1981].

(5-A.38) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold. (1) If G_1 of Σ_1 is surjective, then dom $T_{12}^{min} = X_1$ holds, where dom T_{12}^{min} denotes the domain of T_{12}^{min}.
(2) If H_2 of Σ_2 is injective, then T_{12}^{max} is a partial function : $X_1 \to X_2$.

[proof] This can be proved the same as in Matsuo [1981].

(5-A.39) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, then GrT_{12}^{max} is an invariant sub-product linear U-action of (X_1, F_1) and (X_2, F_2).

[proof] By the definition of GrT_{12}^{max}, $GrT_{12}^{max} = \{(x_1, x_2) \in X_1 \times X_2; H_1 x_1 = H_2 x_2\}$ holds. Let (x_1, x_2) and $(x_1', x_2') \in GrT_{12}^{max}$, i.e., $H_1 x_1 = H_2 x_2$ and $H_1 x_1' = H_2 x_2'$ hold. $H_1(x_1 + x_1') = H_1 x_1 + H_1 x_1' = H_2 x_2 + H_2 x_2' = H_2(x_2 + x_2')$ hold. This implies $(x_1 + x_1', x_2 + x_2') \in GrT_{12}^{max}$. For $k \in K$ and $(x_1, x_2) \in GrT_{12}^{max}$, $(kx_1, kx_2) \in GrT_{12}^{max}$ holds. Moreover, for $u \in U$ and $(x_1, x_2) \in GrT_{12}^{max}$, $H_1 F_1(u) x_1 = S_l(u) H_1 x_1 = S_l(u) H_2 x_2 = H_2 F_2(u) x_2$ hold. Hence, we obtain $(F_1(u) x_1, F_2(u) x_2) \in GrT_{12}^{max}$. Therefore, $GrT_{12}^{max} \subset X_1 \times X_2$ is invariant under $F_1 \times F_2$. Therefore, $(GrT_{12}^{max}, F_1 \times F_2)$ is a linear U-action.

(5-A.40) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, G_1 be surjective and H_2 be injective, then $T_{12}^{min} = T_{12}^{max}$ holds and T_{12} is an Affine Dynamical System morphism : $\Sigma_1 \to \Sigma_2$ by setting $T_{12} = T_{12}^{min}$.

[proof] If G_1 is surjective and H_2 is injective, then Lemma (5-A.38) implies that $T_{12} \in Mor(\Sigma_1, \Sigma_2)$ is unique, $T_{12} \cdot G_1 = G_2$ and $H_2 T_{12} = H_1$ hold. Owing to Lemma (5-A.39), T_{12} is a linear U-morphism : $(X_1, F_1) \to (X_2, F_2)$.

Remark: The uniqueness part of (sophisticated) Realization Theorem (5-A.35) for input response maps is proven by sophisticated Affine Dynamical Systems being canonical and Lemma (5-A.40).

5.4.B Finite Dimensionality

In this part, we will give proofs for theorems, propositions and corollaries stated in section 5.2.

5.4.B.1 Pointed Finite Dimensional Linear U-Actions

In Appendix 5.4.A, the linear U-actions were introduced. In this section we consider those whose state space is finite dimensional. Then it is shown that finite dimensional linear U-actions can be represented by matrix expressions.

(5-B.1) Definition
A linear U-action (X, F) whose X is finite (n) dimensional is said to be a finite dimensional (n dimensional) linear U-action.

In Appendix 5.4.A, we showed that an initial object of any linear U-action with an affine map $((X, F), g)$ is $((V(\Omega^+), \psi), e)$ and the quasi-reachability of $((X, F), g)$ implies surjection of the corresponding affine input map G. In this section we will give a criterion for being quasi-reachable of finite dimensional linear U-actions with an affine map. Introducing the quasi-reachable standard form, we show that it is a representative of linear U-actions with an affine map.

Let $((X, F), g)$ be a linear U-action with an affine map and G be the affine input map corresponding to an affine map g, namely, a linear U-morphism $G : (V(\Omega^+), \psi) \to (X, F)$ which satisfies $g(u) = G(e_u)$.
Let $QR(i)$ be the linear hull of reachable set by input whose length is within i, i.e.,
$QR(i) := \{\sum_i \alpha_i x_i : x_i = \sum_{j=1}^{|\omega|} F(\omega(|\omega|)F(\omega(|\omega| - 1)) \cdots F(\omega(j))g(\omega(j)),$ $\omega \in \Omega_i^+, \alpha_i \in K\}$.
Where $\Omega_i^+ := \{\omega \in \Omega; |\omega| \leq i \in N\}$.
Then the following formula holds.
$QR(i + 1) = QR(i)+ \ll \{F(u)x + g(u); u \in U, x \in QR(i)\} \gg$.
Therefore, the following sequence can be obtained.
$0 = QR(0) \subseteq QR(1) \subseteq \cdots \subseteq QR(i) \subseteq \cdots \subseteq QR(\infty)$.
And $QR(n) = G(\ll \Omega_n^+ \gg)$ holds.
Where, $\ll \Omega_n^+ \gg$ denotes the linear hull of Ω_n^+ in $V(\Omega^+)$.

Moreover, let $G_l = G \cdot J_l$, where J_l is the canonical injection $:\ll \Omega_n^+ \gg\rightarrow V(\Omega^+)$. Then the above sequence can be rewritten as follows:

$$\text{im } G_0 \subseteq \text{im } G_1 \subseteq \cdots \subseteq \text{im } G_i \subseteq \cdots \subseteq \text{im } G_\infty = \text{im } G.$$

Then we can obtain next lemma easily.

(5-B.2) Lemma
If im $G_{j-1} =$ im G_j for an integer $j \in N$, then im $G_j =$ im G_{j+1}.

[proof] By the formula, im $G_j =$ im $G_{j-1}+ \ll \{F(u)x; u \in U, x \in$ im $G_{j-1}\} \gg$ holds. By assumption im $G_{j-1} =$ im G_j, im $G_{j+1} =$ im $G_{j-1}+$ $\ll \{F(u)x; u \in U, x \in$ im $G_{j-1}\} \gg=$ im G_j holds.

(5-B.3) Lemma
For any linear U-action with an affine map $((K^n, F), g)$, then im $G_n =$ im G always holds. Therefore, ((im $G_n, F), g)$ is quasi-reachable linear U-action with an affine map.

[proof] This is a direct consequence of Lemma (5-B.2) and definition of quasi-reachability.

(5-B.4) Proposition
Let $((K^n, F), g)$ be a linear U-action with an affine map, then $((K^n, F), g)$ is quasi-reachable if and only if im $G_n = K^n$ holds.

[proof] The necessary and sufficient condition for being quasi-reachable of $((K^n, F), g)$ is that im $G = K^n$. By Lemma (5-B.3), this is equivalent to im $G_n = K^n$. Consequently, the proposition holds.

(5-B.5) Proposition
Let $((K^n, F), g)$ be a quasi-reachable linear U-action with an affine map, then im G_j is more than j dimensional for any integer $j (1 \leq j \leq n)$.

[proof] For any integer j, let's assume that there does not exist j linearly independent vectors in im G_j. And if im $G_{j-1} \subset$ im G_j holds, then the condition contradicts the nonexistence of j vectors. Hence, im $G_{j-2} =$ im $G_{j-1} = \cdots =$ im G holds and im G has no more than j vectors. This contradicts the quasi-reachability of $((K^n, F), g)$.

(5-B.6) Proposition
Let $((K^n, F), g)$ be a linear U-action with an affine map. $((K^n, F), g)$ is quasi-reachable if and only if
rank $[g, F(u_1)g, F(u_2)g, \cdots, F(u_m)g, F(u_1)^2g, F(u_1)F(u_2)g, F(u_1)F(u_3)g,$
$\cdots, F(u_1)F(u_m)g, F(u_2)^2g, \cdots, F(u_1)^{n-1}g,\ F(u_1)^{n-2}F(u_2)g, \cdots,$
$F(u_m)^{n-1}g] = n$ holds.

[proof] This can be obtained by Proposition (5-B.4).

(5-B.7) Definition
Let $((K^n, F_s), g_s)$ be a quasi-reachable linear U-action with an affine map. If $((K^n, F_s), g_s)$ satisfies the following conditions, then it is said to be the quasi-reachable standard form.
1) $\mathbf{e_1} = g_s(\omega_1)$, $\mathbf{e_i} = \sum_{j=1}^{|\omega_i|} F_s(\omega_i(|\omega_i|)F_s(\omega_i(|\omega_i|-1)) \cdots F_s(\omega_i(j))g_s(\omega_j(j))$ for $\{\omega_i; 1 \leq i \leq n\}$ and $\omega_1 < \omega_2 < \cdots \omega_n$.
2) $\omega_1 < \omega_2 < \cdots \omega_n$ and $|\omega| \leq n-1$ for $i(1 \leq i \leq n)$ hold.
3) $\sum_{j=1}^{|\omega|} F_s(\omega(|\omega|)F_s(\omega(|\omega|-1)) \cdots F_s(\omega(j))\mathbf{e_k} = \sum_{j=1}^{n} \alpha_j \mathbf{e_j}$ holds for any input sequence ω such that $\omega_j < \omega < \omega_{j+1} (1 \leq j \leq n-1)$ and some k such that $|\omega_k| = 1$.

(5-B.8) Proposition
For any quasi-reachable linear U-action with an affine map $((K^n, F), g)$, there uniquely exists the quasi-reachable standard form $((K^n, F_s), g_s)$ which is isomorphic to it.

[proof] We select the set of linearly n independent vectors:
$\{x_i := \sum_{j=1}^{|\omega_i|} F(\omega_i(|\omega_i|)F(\omega_i(|\omega_i|-1)) \cdots F(\omega_i(j))g(\omega_i(j)); 1 \leq i \leq n, \omega_i \in \Omega\}$ in the order of the numerical value of Ω. Then the condition $\omega_i \leq i-1$ for $i(1 \leq i \leq n)$ holds by Proposition (5-B.5). See Definition (5.19) for the numerical value.
We introduce a linear operator $T : K^n \to K^n$ by setting $Tx_i = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_s(u) := TF(u)T^{-1}$, $Tg(u) = g_s(u)$ for any $u \in U$, then $F_s(u) \in K^{n \times n}$ and a collection $((K^n, F_s), g_s)$ is a linear U-action with an affine map. Since $Tx_i = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, the state $\mathbf{e_i}$ is a reachable state by input ω_i whose length is shorter than i. T is a linear U-morphism with an affine map: $((K^n, F), g) \to ((K^n, F_s), g_s)$. T preserves the linear independence and dependence. Therefore, $((K^n, F_s), g_s)$ is a quasi-

reachable standard form. Next, we can easily show that the uniqueness of it comes from the selection of $\{\omega_i; 1 \leq i \leq n\}$.

Remark: There are many equivalences in the category of linear U-actions with an affine map, and this proposition says that the equivalences can be represented as quasi-reachable standard forms.

5.4.B.2 Finite Dimensional Linear U-Actions with a Readout Map

In Appendix 5.4.A, we showed that the final object of any linear U-action with a readout map $((X, F), h)$ is $((F(\Omega, Y), S_l, 1)$ and the distinguishability of $((X, F), h)$ implies injection of the corresponding linear observation map H. In this section we will give a criterion for being distinguishable of finite dimensional linear U-actions with a readout map. Introducing the distinguishable standard form, we show that it is a representative of linear U-actions with a readout map.

Let $((X, F), h)$ be a linear U-action with a readout map and H be the linear observation map corresponding to a readout map h, namely, a linear U-morphism $H : (X, F) \to (F(\Omega, Y), S_l)$ which satisfies $1 \cdot H = h$.

Let $LO(i)$ be the linear hull of reachable set by output whose length is within i, i.e. $LO(i) := \{\sum_j \alpha_j x'_j; x'_j = h\phi_F(\omega_j), \alpha_j \in K, \omega_j \in \Omega_i\}$.
Where $\Omega_i := \{\omega; |\omega| \leq i\}$, then the following sequence holds:
$LO(0) \subseteq LO(1) \subseteq \cdots LO(i) \subseteq \cdots \subseteq LO(\infty)$.

Let $H_l = P_l \cdot H$, where P_l is the canonical surjection: $F(\Omega, Y) \to F(\Omega_l, Y)$, and $F(\Omega_l, Y) := \{a \in F(\Omega, Y); a : \Omega_l \to Y\}$.
Then ker $H_l = LO(l)^0$ holds, i.e., ker $H_l = \{x \in X; hx = 0 \text{ for } h \in LO(l)\}$.
Moreover, ker $H = LO(\infty)^0$ holds.

(5-B.9) Lemma
For any linear U-action with a readout map $((X, F), h)$, $LO(n-1) = \ll h\phi_F(\Omega) \gg$ holds. Where $h\phi_F(\Omega) = \{h\phi_F(\omega); \omega \in \Omega\}$.

[proof] This can be obtained the same way as Lemma (5-B.3).

(5-B.10) Proposition
For any linear U-action with a readout map $((X, F), h)$, ((ker $H_{n-1}, F)$ is a

sub linear U-action of (X, F) and $((K^n/\ker\ H_{n-1}, \tilde{F}), \tilde{h})$ is a distinguishable linear U-action with a readout map.

[proof] Let H be the corresponding linear observation map to h. By Lemma (5-B.9), $LO(n-1) = \ll hF(\Omega) \gg$ holds. Therefore, $\ker\ H_{n-1} = \ker\ H$ holds. Because H is a linear U-morphism : $(K^n, F) \to (F(\Omega, Y), S_l)$, $(\ker\ H_{n-1}, F)$ is a sub linear U-action of (K^n, F).
Therefore, $((K^n/\ker\ H_{n-1}, \tilde{F}), \tilde{h})$ can be introduced, and become a distinguishable linear U-action with a readout map.

(5-B.11) Proposition
Let $((X, F), h)$ be a linear U-action with a readout map. $((X, F), h)$ is distinguishable if and only if $LO(n-1) = K^{p\times n}$ holds.

[proof] This can be obtained the same as Proposition (5-B.4).

(5-B.12) Proposition
If $((X, F), h)$ is distinguishable, then $LO(j-1)$ is more than j dimensional for any $j (1 \leq j \leq n)$.

[proof] This can be obtained the same as Proposition (5-B.5).

(5-B.13) Proposition
Let $((X, F), h)$ be a linear U-action with a readout map. $((X, F), h)$ is distinguishable if and only if
rank $[h^T, (hF(u_1))^T, \cdots, (hF(u_m))^T, (hF(u_1)^2)^T, (hF(u_1)F(u_2))^T, \cdots,$
$(hF(u_1)^{n-1})^T, \cdots, (hF(u_m)^{n-1})^T] = n$ holds.
Where T denotes the transpose of matrix.

[proof] This can be obtained the same as Proposition (5-B.6).

(5-B.14) Definition
Let $((X, F_s), h_s)$ be a distinguishable linear U-action with a readout map. If $((X, F_s), h_s)$ satisfies the following conditions, then it is said to be the distinguishable standard form:
1) $\mathbf{e_i}^T = h(\sum_{j=1}^{|\omega_i|} F(\omega_i(|\omega_i|))F(\omega_i(|\omega_i|-1)) \cdots F(\omega_i(j)))$ holds for input sequences $\{\omega_i; 1 \leq i \leq n\}$.
2) $1 = \omega_1 < \omega_2 < \cdots < \omega_n$ and $|\omega_i| \leq i-1$ for $i (1 \leq i \leq n)$ hold.

3) $h(\sum_{j=1}^{|\omega|} F(\omega(|\omega|))F(\omega(|\omega|-1))\cdots F(\omega(j))) = \sum_{i=1}^{j} \alpha_j \mathbf{e_i}$ holds for any input sequence ω such that $\omega_j < \omega < \omega_{j+1} (1 \le j \le n-1)$.

Remark: If $((X, F_s), h_s)$ is the distinguishable standard form, note that $h_s = \mathbf{e_1}^T$.

(5-B.15) Propositon
For any distinguishable linear U-action with a readout map $((X, F), h)$, there exists uniquely the distinguishable standard form $((X, F_s), h_s)$ which is isomorphic to it.

[proof] We select the set of linearly n independent vectors as the following:
$\{h(\sum_{j=1}^{|\omega_i|} F(\omega_i(|\omega_i|)F(\omega_i(|\omega_i|-1))\cdots F(\omega_i(j))); 1 \le i \le n, \omega_i \in \Omega\}$ in the order of index value of Ω. Then the condition $|\omega_i| \le i-1$ for $i(1 \le i \le n)$ holds by Proposition (5-B.12). We introduce a linear operator $T : K^n \to K^n$ by setting $h(\sum_{j=1}^{|\omega_i|} F(\omega_i(|\omega_i|)F(\omega_i(|\omega_i|-1))\cdots F(\omega_i(j)))^T = \mathbf{e_i}^T$, for $i(1 \le i \le n)$, then T is a regular matrix. Let $F_s(u) := TF(u)T^{-1}$ for any $u \in U$, then $F_s(u) \in K^{n \times n}$ and a collection $((X, F_s), \mathbf{e_1}^T)$ is a linear U-action with a readout map. T is a linear U-morphism with a readout map : $((K^n, F), h) \to ((K^n, F_s), \mathbf{e_1}^T)$. T preserves the linear independence and dependence. Therefore, $((K^n, F_d), \mathbf{e_1}^T)$ is the distinguishable standard form. Next, we can show the uniqueness of it comes from the selection of $\{\omega_i; 1 \le i \le n\}$.

Remark: There are many equivalences in the category of linear U-actions with a readout map, and this proposition says that the equivalences can be represented as the distinguishable standard form.

5.4.B.3 Finite Dimensional Affine Dynamical Systems

This section is prepared for the proofs of Representation Theorem (5.21) and (5.23) for finite dimensional canonical Affine Dynamical Systems.

(5-B.16) Proof of Representation Theorem (5.21)
Note that the pointed linear U-action in the quasi-reachable standard system is the quasi-reachable standard form. Let $\sigma = ((K^n, F), g, h, h^0)$ be any finite dimensional canonical Affine Dynamical System. For the quasi-

reachable standard form $((K^n, F_s), g_s)$ and a linear U-morphism with an affine map $T : ((K^n, F), g) \to ((K^n, F_s), g_s)$ introduced in the proof of Proposition (5-B.8), let $h_s := h \cdot T^{-1}$. Then T is a Linear Representation System morphism : $\sigma = ((K^n, F), g, h, h^0) \to \sigma_s = ((K^n, F_s), g_s, h_s, h^0)$. And T is bijective and σ_s is the only quasi-reachable standard system. By Corollary (5.4), the behaviors of σ and σ_s are the same.

(5-B.17) Proof of Representation Theorem (5.23)
Note that a linear U-action with a readout map in the quasi-reachable standard system is the distinguishable standard form.
Let $\sigma = ((K^n, F), g, h, h^0)$ be any finite dimensional canonical Affine Dynamical System. For the distinguishable standard form $((K^n, F_s), \mathbf{e_1}^T)$ and a linear U-morphism with a readout map $T : ((K^n, F), h) \to ((K^n, F_s), \mathbf{e_1}^T)$ introduced in the proof of Proposition (5-B.15), let $g_s := Tg$. Then T is a linear Affine Dynamical System morphism : $\sigma = ((K^n, F), g, h, h^0) \to \sigma_s = ((K^n, F_s), g_s, h_s, h^0)$. And T is bijective and σ_s is the only distinguishable standard system. By Corollary (5.4), the behaviors of σ and σ_s are the same.

5.4.B.4 Existence Criterion for Affine Dynamical Systems

This section is prepared for the proofs of the theorem for existence criterion (5.25).
Let $G_l = G \cdot J_l$, where J_l is the canonical injection $:\ll \Omega_l^+ \gg\to V(\Omega^+)$.
Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(\Omega, Y) \to F(\Omega_l, Y)$,

(5-B.18) Proof of Theorem (5.25)
Let A be the affine input/output map corresponding to input response map $a \in F(\Omega, Y)$. Obviously, im $A = \ll \{S_l(\omega)a - a : \omega \in \Omega\} \gg$. Let $A_l := A \cdot J_l$, and let a linear operator $A_{(l,m)} :\ll \Omega_l^+ \gg\to F(\Omega_m, Y)$ be defined by setting $A_{(l,m)} := P_m \cdot A \cdot J_l$, then $A_{(l,m)}$ can be represented by a partial Hankel matrix $H^A_{a\ (l,m)}$ of the Hankel matrix H^A_a.
Where $H^A_{a\ (l,m)} = [a(\bar{\omega}|\omega) - a(\bar{\omega})]$ for $\omega \in \Omega_l$, $\bar{\omega} \in \Omega_m$. And $\Omega_m = \{\omega \in \Omega : |\omega| \le m\}$ for some integer m.
First, we show 1) $\Longrightarrow$ 2). By Theorem (5.14) and Corollary (5-A.36), im A is n dimensional. If im $A_n \ne$ im A_{n+1}, then the dimension of im A_n is $n+1$ or more by Lemma (5-B.2), therefore, im $A_n =$ im $A_{n+1} = \cdots =$ im A. Consequently, there exist n linearly independent vectors in $\{S_l(\omega)a - a; |\omega| \le n$ for $\omega \in \Omega\}$, but not $n+1$ or more linearly independent vectors in it.

Secondly, we show 2) $\Longrightarrow$ 3). Since im $A_n =$ im A_{n+1}, im $A_n =$ im $A_{n+1} = \cdots =$ im A holds. Therefore, the dimension of im A_r is n for $r \geq n$. On the other hand, by Corollary (5-A.36) and Lemma (5-B.10), ker $P_s = 0$ for $s \geq n-1$. Consequently, the dimension of im $P_s \cdot A \cdot J_r$ is n. Therefore, the rank of partial Hankel matrix $H^A_{a\ (r,s)}$ corresponding to $P_s A J_r$ is n.
Lastly, we show 3) $\Longrightarrow$ 1). Since the rank of the Hankel matrix H^A_a is n, the range im A of the affine input/output map A corresponding to H^A_a is n dimensional. By im $A = \ll \{S_l(\omega)a - a; \omega \in \Omega\} \gg$ and Corollary (5-A.36), 1) is obtained.

5.4.B.5 Realization Procedure for Affine Dynamical Systems

This section is prepared for the proof of Theorem for Realization Procedure (5.26).

(5-B.19) Proof of Theorem (5.26)
Let $S_l(\Omega)a - a := \{S_l(\omega)a - a; \omega \in \Omega\}$. By Theorem (5.14), $((\ll S_l(\Omega)a - a \gg, S_l), \xi, \mathit{1}, a(\mathit{1}))$ is a canonical Affine Dynamical System that realizes $a \in F(\Omega, Y)$. The linearly independent vectors $\{S_l(\omega i)a - a; \omega_1 < \omega_2 < \cdots < \omega_n, \omega_i \in \Omega$ and $|\omega_i| \leq i$ for $i (1 \leq i \leq n)\}$ satisfies $\ll \{S_l(\omega i)a - a; \omega_1 < \omega_2 < \cdots < \omega_n, \omega_i \in \Omega$ and $|\omega_i| \leq i$ for $i(1 \leq i \leq n)\} \gg = \ll S_l(\Omega)a - a \gg$. Let a linear map $T : \ll S_l(\Omega)a - a \gg \rightarrow K^n$ be $T \cdot (S_l(\omega_i)a - a) = \mathbf{e_i}$ for any $i(1 \leq i \leq n)$. Then, by step 3), $h_s \cdot T = 1$ holds. And by step 4), $F_s(u) \cdot T = T \cdot F_s(u)$ holds for any $u \in U$. Consequently, T is bijective and an Affine Dynamical System morphism : $((\ll S_l(\Omega)a - a \gg, S_l), \xi, 1, a(\mathit{1})) \rightarrow \sigma_s = ((K^n, F_s), g_s, h_s, a(\mathit{1}))$. By Corollary (5.4), the behavior of σ_s is a. It follows from the choice of $\{S_l(\omega i)a - a; \omega_1 < \omega_2 < \cdots < \omega_n, \omega_i \in \Omega$ and $|\omega_i| \leq i$ for $i(1 \leq i \leq n)\}$ and the determination of map T that σ_s is the quasi-reachable standard system.

6 Pseudo Linear Systems

Pseudo Linear Systems are presented with the following main theorem. The main theorem says that for any input/output map with causality and time-invariance, there exist at least two canonical Pseudo Linear Systems which realize (faithfully describe) it and any two canonical Pseudo Linear Systems with the same behavior are isomorphic.

Secondly, details of finite dimensional Pseudo Linear Systems are investigated. A criterion for canonical finite dimensional Pseudo Linear Systems is given. A representation theorem of isomorphic classes of canonical Pseudo Linear Systems is given. A criterion for the behavior of finite dimensional Pseudo Linear Systems is given by rank condition of Input/Output Matrix. Also a procedure to obtain a canonical Pseudo Linear System is given.

Thirdly, partial realization of them will be discussed according to the above results. Existence of minimum partial realization is easily presented. It hardly ever happens for minimum partial realizations to be unique up to isomorphism. To solve the uniqueness problem, we introduce the notion of natural partial realizations.

The main results for partial realization are the following:
1) A necessary and sufficient condition for the existence of the natural partial realizations is given by the rank condition of finite sized Input/Output Matrix.
2) The existence condition of natural partial realization is equivalent to the uniqueness condition of minimum partial realizations.
3) An algorithm to obtain a natural partial realization from a partial time-invariant input response map is given.

Moreover, for the time-invariant input response map, we can discuss a real time partial realization problem. Namely, by a single experiment, we find a mathematical model from on-line data. An algorithm to obtain a partial realization from the data is given if a physical object is finite dimensional.

6.1 Input Response Maps with Time-Invariance

In this chapter we consider input/output maps $a \in F(\Omega, Y)$ which satisfy the following time-invariant condition. It is said to be a time-invariant input response map. Where Y is a linear space over the field K.

T. Matsuo, Y. Hasegawa: Realization Theory of Discrete-Time Dynamical Systems, LNCIS 296, pp. 113–155, 2003.

(6.1) Definition

If an input response map $a \in F(\Omega, Y)$ satisfies the following time-invariant condition, then a is said to be a time-invariant input response map.
Time-invariant condition : $a(\omega_1|\omega) - a(\omega 1) = a(\bar{\omega}_1|\omega) - a(\bar{\omega}_1)$ for any $\omega \in \Omega$, and $\omega_1, \bar{\omega}_1 \in \Omega$ such that $|\omega_1| = |\bar{\omega}_1|$.

Remark: The word "time-invariant" comes from Remark in Corollary (6-A.24).

(6.2) Definition

For any time-invariant input response map $a \in F(\Omega, Y)$, a function $GI_a : U \to F(\Omega, Y); u \mapsto GI_a(u)[; t \mapsto a(u^t) - a(u^{t-1})]$ is said to be a modified impulse response of a.
Where u^t is by $u^t(i) = u$ for $i(1 \leq i \leq t)$.

(6.3) Representation Theorem

For any time-invariant input response map $a \in F(\Omega, Y)$, there exists uniquely the modified impulse response of a by the following equation. This correspondence is bijective.

$$a(\omega) = a(1) + \sum_{j=1}^{|\omega|}\{((GI_a)(\omega(j)))(|\omega| - j)\}.$$

[proof] This theorem is obtained by direct calculation.

6.2 Pseudo Linear Systems

(6.4) Definition

A system given by the following equations is written as a collection $\sigma = ((X, F), g, h, h^0)$ and it is said to be a Pseudo Linear System.

$$\begin{cases} x(t+1) &= Fx(t) + g(\omega(t+1)) \\ x(0) &= 0 \\ \gamma(t) &= h^0 + hx(t) \end{cases}$$

Where X is a linear space over the field K, F is a linear operator on X and $\omega(t) \in U$ for any $t \in N$. And g is a function : $U \mapsto X$, h is a linear operator : $X \to Y$ and $h^0 \in Y$.
The input response map $a_\sigma \in F(\Omega, Y); \omega \mapsto h^0 + h(\sum_{j=1}^{|\omega|}\{((F^{|\omega|-j})g(\omega(j)))$ is said to be a behavior of σ.

For a time-invariant input response map $a \in F(\Omega, Y)$, σ that satisfies $a_\sigma = a$ is called a realization of a.
Note that the behavior a_σ of a Pseudo Linear System σ is a time-invariant input response map.
A Pseudo Linear System σ is said to be quasi-reachable if the linear hull of the reachable set $\{\sum_{j=1}^{|\omega|}\{((F^{|\omega|-j})g(\omega(j))); \omega \in \Omega\}$ is equal to X. A Pseudo Linear System σ is called observable if $hF^m x_1 = hF^m x_2$ for any $m \in N$ implies $x_1 = x_2$.
A Pseudo Linear System σ is said to be canonical if σ is quasi-reachable and observable.

Remark 1: The $x(t)$ in the system equation of σ is the state that produces output values of a_σ at the time t by adding h^0, namely the state $x(t)$ and linear operator $h : X \to Y$ generates the output value $a_\sigma(t) = h^0 + hx(t)$.
Remark 2: It is meant for σ to be a faithful model for the time-invariant input response map $a \in F(\Omega, Y)$ such that σ realizes a.
Remark 3: Notice that a canonical Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ is a system that has the most reduced state space X among systems that have the behavior a_σ (see Corollary (6-A.16), Proposition (6-A.23), Corollary (6-A.24), Propositions (6-A.25), (6-A.29) and (6-A.30), Definition (6-A.31) and Proposition (6-A.33) in Appendix 6.7).

(6.5) Example
$A(N \times U, K) := \{\lambda = \sum_{n,u} \lambda(n, u)\mathbf{e}_{(\mathbf{n},\mathbf{u})}(\text{finite sum}); n \in N, u \in U\}$. Where $\mathbf{e}_{(\mathbf{n},\mathbf{u})}$ is given by the following equations for $n, n' \in N$ and $u, u' \in U$. If $n = n'$ and $u = u'$ implies $\mathbf{e}_{(\mathbf{n},\mathbf{u})}(n', u') = 1$. If $n \neq n'$ or $u \neq u'$ implies $\mathbf{e}_{(\mathbf{n},\mathbf{u})}(n', u') = 0$. Then $A(N \times U, K)$ is clearly a linear space. Let S_r be $S_r(\mathbf{e}_{(\mathbf{n},\mathbf{u})}) = S_r(\mathbf{e}_{(\mathbf{n+1},\mathbf{u})})$, then $S_r \in L(A(N \times U, K))$ and S_r is irrelevant to the input value's set U. S_r is a right shift operator. Let a map $\eta : U \to A(N \times U, K); u \mapsto \mathbf{e}_{(\mathbf{0},\mathbf{u})}$ and let a linear map $\bar{a} : A(N \times U, K) \to Y$ be $\bar{a}(\mathbf{e}_{(\mathbf{n},\mathbf{u})}) = a(u^{n+1}) - a(u^n)$ for any time-invariant input response map $a \in F(\Omega, Y)$. Then a collection $((A(N \times U, K), S_r), \eta, \bar{a}, a(\mathit{1}))$ is a quasi-reachable Pseudo Linear System that realizes a.
Let $F(N, Y) := \{$ any function $f : N \to Y\}$. Let $S_l\ \gamma(t) = \gamma(t + 1)$ for any $\gamma \in F(N, Y)$ and $t \in N$, then $S_l \in L(F(N, Y))$. Let a map $\chi : U \to F(N, Y)$ be $(\chi(u))(t) := a(\omega|u) - a(\omega)$ for any $u \in U, t \in N$, a time-invariant input response map $a \in F(\Omega, Y)$ and ω such that $|\omega| = t$. Moreover, let a linear map 0 be $F(N, Y) \to Y; \gamma \mapsto \gamma(0)$. Then a collection $((F(N, Y), S_l), \chi, 0, a(\mathit{1}))$ is a distinguishable Pseudo Linear System that realizes a.

(6.6) Theorem
The following two Pseudo Linear Systems are canonical realizations of any time-invariant input response map $a \in F(\Omega, Y)$.
1) $((A(N \times U, K)/_{=a}, \tilde{S}_r), \tilde{\eta}, \tilde{\bar{a}}, a(1))$.
Where $A(N \times U, K)/_{=a}$ is a quotient space obtained by equivalence relation $\sum_{n,u} \lambda_1(n,u)\mathbf{e_{(n,u)}} = \sum_{\bar{n},\bar{u}} \lambda_2(\bar{n},\bar{u})\mathbf{e_{(\bar{n},\bar{u})}} \Longleftrightarrow \sum_{n,u}(a(u^{n+1} - a(u^n)) = \sum_{\bar{n},\bar{u}}(a(\bar{u}^{\bar{n}+1} - a(\bar{u}^{\bar{n}}))$.
And $\tilde{S}_r \in L(A(N \times U, K)/_{=a})$ is given by $\tilde{S}_r[\mathbf{e_{(n,u)}}] = [\mathbf{e_{(n+1,u)}}]$ for $[\mathbf{e_{(n,u)}}] \in A(N \times U, K)/_{=a}$, and $\tilde{\eta}$ is a map : $U \to A(N \times U, K)/_{=a}; u \mapsto [\mathbf{e_{(0,u)}}]$, $\tilde{\bar{a}}$ is given by : $\tilde{\bar{a}} \to Y; [\mathbf{e_{(n,u)}}] \mapsto a(u^{n+1}) - a(u^n)$.
2) $((\ll S_l^N(\chi(U)) \gg, S_l), \chi, 0, a(1))$.
Where $\ll S_l^N(\chi(U)) \gg$ is the smallest linear space which contains $S_l^N(\chi(U))$ $:= \{S_l^i(\chi(u)); u \in U, i \in N, S_l^i(\chi(u))(t) = (\chi(u))(t+i) = a(\omega|u) - a(\omega), \omega \in \Omega, |\omega| = t+i\}$.

[proof] See Corollary (6-A.24), Proposition (6-A.25), Example (6-A.32), Proposition (6-A.30), Proposition (6-A.33) and Corollary (6-A.35).

(6.7) Definition
Let $\sigma_1 = ((X_1, F_1, g_1, h_1, h^0)$ and $\sigma_2 = ((X_2, F_2, g_2, h_2, h^0)$ be Pseudo Linear Systems, then a linear operator $T : X_1 \to X_2$ is said to be a Pseudo Linear System morphism $T : \sigma_1 \to \sigma_2$ if T satisfies $TF_1 = F_2T$, $Tg_1 = g_2$ and $h_1 = h_2T$.
If $T : X_1 \to X_2$ is bijective, then $T : \sigma_1 \to \sigma_2$ is said to be an isomorphism.

(6.8) Realization Theorem of Pseudo Linear Systems
For any time-invariant input response map $a \in F(\Omega, Y)$, there exist at least two canonical Pseudo Linear Systems which realize a. (Existence part)
Let σ_1 and σ_2 be any two canonical Pseudo Linear Systems that realize a time-invariant input response map $a \in F(\Omega, Y)$, then there exists an isomorphism $T : \sigma_1 \to \sigma_2$. (Uniqueness part)

[proof] The first half is obvious from Theorem (6.6). The latter part is obtained by Corollary (6-A.24), Propositions (6-A.25), (6-A.30) and (6-A.33) and Theorem (6-A.34).

6.3 Finite Dimensional Pseudo Linear Systems

Based on the realization theory (6.8), we study structures of finite-dimensional Pseudo Linear Systems in this section.

To obtain concrete results, we assume that the set U of input values is finite, i.e., $U := \{u_i; 1 \leq i \leq m\}$ for some $m \in N\}$. This assumption will imply that the g of a Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ is completely determined by the finite vectors $\{g(u_i); 1 \leq i \leq m, m \in N\}$, it is presented that the assumption is not so special.

Main results can be stated in the following four steps:
First, we present conditions when the finite dimensional Pseudo Linear System is canonical.
Secondly, we obtain the representation theorem for finite dimensional canonical Pseudo Linear Systems, namely, we show that there exist two standard systems as representatives in their equivalence classes. One is the quasi-reachable standard system, and the other is the observable standard system.
Thirdly, we give a criterion for the behavior of finite dimensional Pseudo Linear Systems. It is the rank condition of an infinite Input/Output Matrix.
Lastly, we give a procedure to obtain the quasi-reachable standard system that realizes a given time-invariant input response map.
We will prove the above statements in Appendix 6.7.

(6.9) Corollary
Let T be a Pseudo Linear System morphism $T : \sigma_1 \to \sigma_2$, then $a_{\sigma_1} = a_{\sigma_2}$ holds.

[proof] This is a direct calculation by the definition of the behavior and Pseudo Linear System morphism.

Following is a fact about finite dimensional linear spaces:
FACT : < An n-dimensional linear space over the field K is isomorphic to K^n and $L(K^n, K^m)$ is isomorphic to $K^{m \times n}$. (See Halmos [1958]). >
Therefore, without loss of generality, we can consider n-dimensional Pseudo Linear System as $\sigma = ((K^n, F), g, h, h^0)$, where $F \in K^{n \times n}$, $g(u) \in K^n$ and $h \in K^{p \times n}$.

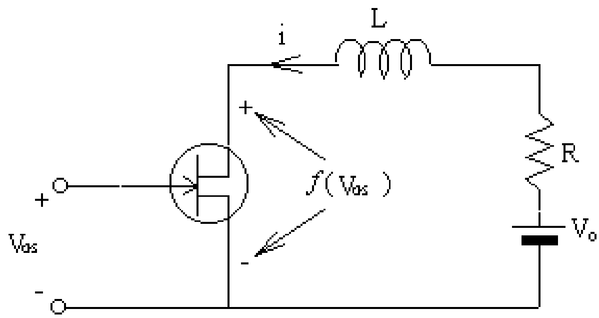


Fig. 6.1. A circuit with the N-channel FET

(6.10) $U = \{u_1, u_2\}$
In this case, a Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ can be completely determined by $\{g(u_i); u_i \in U \text{ for } i = 1, 2\}$.
If on-off inputs are applied to a black-box with time-invariance, the system can be treated in this case. Moreover, if an optimal solution is a bang-bang control when a controlled object with time-invariance is in the optimal controlled condition, then it can be treated in this case.

(6.11) A circuit with the N-channel FET
A circuit shown in Figure 6.1 is represented by the following equation:

$$L\frac{d\ i(t)}{d\ t} - R\ i(t) + f(V_{GS}) = 0$$

Where V_{GS} is the gate-to-source voltage and $f(V_{GS})$ is the drain-to-source voltage. The characteristics of them are shown in Figure 6.1 and 6.2.

Let V_{GS} be an input $\omega(t)$ and the current $i(t)$ be an output $\gamma(t)$. Changing continuous-time to discrete-time by backward difference, we obtain the following equation.

$$\begin{cases} i(t+1) & = & Fi(t) + g(\omega(t+1)) \\ \gamma(t) & = & i(t) \end{cases}$$

Where $F := L/(L + \Delta t \cdot R)$, $g(V_{GS}) := \Delta t(V_0 - f(V_{GS}))/(L + \Delta t \cdot R)$ and Δt is a sampling time.

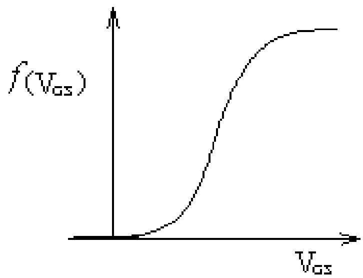


Fig. 6.2. A characteristic of the N-channel FET

Then an affine map g is a map : $U \to R$. Where $U = R$.
Let $g : U \to R$ be a piecewise-linear function with the break points $\{u_i; 1 \leq i \leq m\}$. It follows that an affine map g is completely determined by a finite set $\{g(u_i); 1 \leq i \leq m\}$.

(6.12) Theorem
A Pseudo Linear System $\sigma = ((K^n, F), g, h, h^0)$ is canonical if and only if the following conditions 1) and 2) hold:
1) rank $[g(u_1), Fg(u_1), \cdots, F^{n-1}g(u_1), g(u_2), Fg(u_2), \cdots, F^{n-1}g(u_2), \cdots, g(u_m), Fg(u_m), \cdots, F^{n-1}g(u_m)] = n$
2) rank $[h^T, (hF)^T, \cdots, \cdots, (hF\ n-1)^T] = n$.

[proof] See Proposition (6-B.6) and (6-B.13) in Appendix 6.7.

(6.13) Definition
Let the input value's set U be $U := \{u_i; 1 \leq i \leq m\}$ and let a map $\| \|: N \times U \to N$ be : $(i, u_j) \mapsto \| (i, u_j) \| = m \times i + j$. Then $\| (i, u_j) \|$ is said to be a numerical value of $(i, u_j) \in N \times U$. And we define totally ordered relation by this numerical value in $N \times U$. Namely, $(p, u_p) < (q, u_q) \Longleftrightarrow \| (p, u_p) \| < \| (q, u_q) \|$.

(6.14) Definition
A canonical Pseudo Linear System $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ is said to be a quasi-reachable standard system if a set $\{(I_i, u_{Ji}) \in N \times U, 1 \leq i \leq m\}$ given by $\mathbf{e_i} = F_s^{I_i} g_s(u_{J_i})$ satisfies the following conditions:

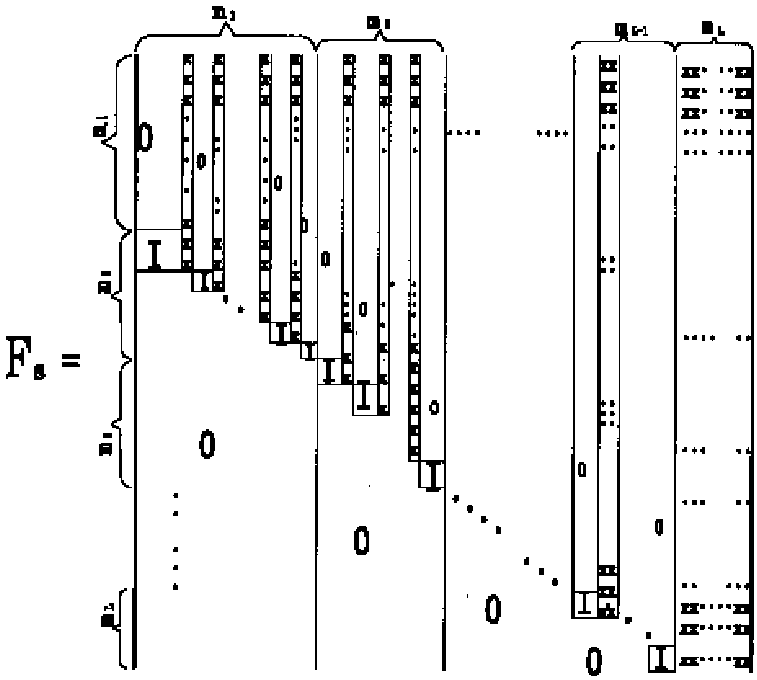


Fig. 6.3. F_s of the quasi-reachable standard system $\sigma_s = ((K^n, F_s, g_s, h_s, h^0)$ defined in (6.14)

1) $(I_1, u_{J_1}) < (I_2, u_{J_2}) < \cdots < (I_n, u_{J_n})$ holds.
2) $I_i < i$
3) $F_s^p g_s(u_q) = \sum_{i=1}^{L} \alpha_i \mathbf{e_i}$ holds for any $(p, u_q) \in N \times U$ such that $(I_j, u_{J_j}) < (p, u_q) < (I_{j+1}, u_{J_{j+1}})$.
Where $\alpha_i \in K$ and $\mathbf{e_i} = [0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T$.

See Figure 6.3 for the quasi-reachable standard system.
Where I in the figure denotes Identity matrix with suitable size and m_j denotes the number of u_{J_j} in the set of the numerical value $\{(I_j, u_{J_j}); I_i = j - 1\}$. And $n = \sum_{i=1}^{j} m_i$ holds.

(6.15) Representation Theorem for equivalence classes
For any finite dimensional canonical Pseudo Linear System, there exists a uniquely determined isomorphic quasi-reachable standard system.

[proof] See (6-B.16) in Appendix 6.7.

(6.16) Definition
A canonical Pseudo Linear System $\sigma_o = ((K^n, F_o), g_o, h_o, h^0)$ is said to be an observable standard system if $\mathbf{e_i}^T = h_o F_o^{i-1}$ holds for $1 \le i \le n$.

(6.17) Representation Theorem for equivalence classes
For any finite dimensional canonical Pseudo Linear System, there exists a uniquely determined isomorphic observable standard system.

[proof] See (6-B.17) in Appendix 6.7.

(6.18) Definition
For any time-invariant input response map $a \in F(\Omega, Y)$, the corresponding linear input/output map $A : ((A(N \times U, K), S_r) \to (F(N, Y), S_l)$ satisfies $A(\mathbf{e_{(s,u)}})(t) = a(u^{s+t+1}) - a(u^{s+t})$.
Therefore, the A can be represented by the next infinite matrix $(I/O)_a$. This $(I/O)_a$ is said to be an Input/Output Matrix of a.

$$
\begin{array}{c} (I/O)_a = \\ t \end{array}
\begin{pmatrix}
 & & \overset{(s,u)}{\vdots} & \\
 & & \vdots & \\
 & & \vdots & \\
\cdots & \cdots & a(u^{s+t+1}) - a(u^{s+t}) & \\
 & & &
\end{pmatrix}
$$

See Corollary (6-A.24) about the corresponding linear input/output map A.

(6.19) Theorem for existence criterion
For a time-invariant input response map $a \in F(\Omega, Y)$, the following conditions are equivalent:
1) The time-invariant input response map $a \in F(\Omega, Y)$ has the behavior of n-dimensional canonical Pseudo Linear System.
2) There exist n linearly independent vectors and no more than n linearly independent vectors in a set $\{S_l^i(\chi(u)); u \in U, i \in N, 1 \le i \le n\}$.
3) The rank of the Input/Output Matrix $(I/O)_a$ of a is n.

[proof] See (6-B.18) in Appendix 6.7.

(6.20) Theorem for a realization procedure
Let a time-invariant input response map $a \in F(\Omega, Y)$ satisfy the condition of Theorem (6.19), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ which realizes a can be obtained by the following procedure:
1) Select the linearly independent vectors $\{S_l^{I_i}(\chi(u_{J_i}))\}$ of the set $\{S_l^i(\chi(u)); u \in U, i \in N, 0 \le i \le n-1\}$ in order of the numerical value. Let $n :=$ rank H_a.
2) Let the state space be K^n. Let the map $g_s : U \to K^n$ be $g_s(u_{J_i}) := \mathbf{e_i}$ for $u_{J_i} \in U$ such that $I_i = 0$. And let $g_s(u_j) := \sum_{i=1}^{j} \alpha_j \mathbf{e_j}$ for $u_j \in U$ and $\chi(u_j) := \sum_{i=1}^{j} \alpha_i \chi(u_{I_i})$ such that $(0, u_{J_i}) < (0, u_j) < (0, u_{J_{i+1}})(or(1, u_{J_{i+1}}))$.
3) Let the output map $h_s = [a(u_{J_1}^{I_1+1}) - a(u_{J_1}^{I_1}), a(u_{J_2}^{I_2+1}) - a(u_{J_2}^{I_2}), \cdots, a(u_{J_n}^{I_n+1}) - a(u_{J_n}^{I_n})]$
4) Let f_j in $F_s := [f_1, f_2, \cdots, f_n]$ be $f_j := [f_{i,1}, f_{i,2}, \cdots, f_{i,n}]^T$.
Where $S_l^{I_i+1}\chi(u_{J_i}) = f_{i,j} S_l^{I_j}\chi(u_{J_j})$, $f_{i,j} \in K$.

[proof] See (6-B.19) in Appendix 6.7.

6.4 Partial Realization Theory of Pseudo Linear Systems

Here we consider a partial realization problem by multi-experiment. Let $\underline{a}$ be an $\underline{N}$ sized time-invariant input response map ($\in F(\Omega_{\underline{N}}, Y)$, where $\underline{N} \in N$ and $\Omega_{\underline{N}} := \{\omega \in \Omega; |\omega| \le \underline{N}\}$. The $\underline{a}$ is said to be a partial time-invariant input response map.

A finite dimensional Pseudo Linear System $\sigma = ((X, F), g, h.x^0)$ is said to be a partial realization of $\underline{a}$ if $h^0 + h(\sum_{j=1}^{|\omega|} F^{|\omega|-j} g(\omega(j))) = \underline{a}(\omega)$ holds for any $\omega \in \Omega_{\underline{N}}$.

A partial realization problem of Pseudo Linear Systems can be stated as follows:
< For any given partial time-invariant input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, find a partial realization σ of $\underline{a}$ such that the dimensions of state space X of σ is minimum, where the σ is said to be a minimal partial realization of $\underline{a}$. Moreover, show when the minimal realizations are isomorphic.>

In section 6.1, we have obtained the representation theorem for the time-invariant input response maps. The theorem says that any time-invariant input response map can be characterized by the modified impulse response. Note that the modified impulse response $GI : U \to F(N, Y)$ can be represented by $(GI(u)(t)) = a(u^{t+1}) - a(u^t)$ for $u \in U, t \in N$ and the time-invariant input response map $a \in F(\Omega, Y)$.

For any given partial time-invariant input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, this correspondence can determine a partial modified impulse response $\underline{GI} : U \to F(N_{\underline{N}-1}, Y)$. Where $N_{\underline{N}-1} := \{1, 2, , \underline{N} - 1$; for some $\underline{N} \in N$.

(6.21) Proposition

For any given time-invariant input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, there always exists a minimal partial realization of it.

[proof] For any $\omega \notin \Omega_{\underline{N}}$, set $\underline{a}(\omega) = 0$. Then $\underline{a} \in F(\Omega, Y)$, and Theorem (6.19) implies that there exists a finite dimensional partial realization of $\underline{a}$. Therefore, there exists a minimal partial realization of it.

Minimal partial realizations are generally not unique modulo isomorphism. Therefore, we introduce a natural partial realization, and we show that natural partial realizations exist if and only if they are isomorphic.

(6.22) Definition

For a Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ and some $p \in N$, if $X = \ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j} g(\omega(j))); \omega \in \Omega_p\} \gg$, then σ is said to be p-quasi-reachable.

Let q be some integer. If $hF^{q'}x = 0$ for any $q' \leq q$ implies $x = 0$, then σ is said to be q-observable.

For a given time-invariant input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, if there exist p and $q \in N$ such that $p + q < \underline{N}$ and σ is p-quasi-reachable and q-observable then σ is said to be a natural partial realization of $\underline{a}$.

For a partial time-invariant input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, the following matrix $(I/O)_{\underline{a}\ (p, \underline{N}-p)}$ is said to be a finite-sized Input/Output Matrix of $\underline{a}$.

$$
(I/O)_{\underline{a}\ (p,\underline{N}-p)} = \begin{pmatrix} & & (s,u) \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ t & \cdots \quad \cdots & a(u^{s+t+1}) - a(u^{s+t}) \\ & & \end{pmatrix}
$$

Where $0 \leq s \leq p, 0 \leq t \leq \underline{N} - p$ and $u \in U$.

(6.23) Theorem
Let $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ be the finite Input/Output Matrix of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then there exists a natural partial realization of $\underline{a}$ if and only if the following conditions hold:
rank $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ = rank $(I/O)_{\underline{a}\ (p,\underline{N}-p-1)}$ = rank $(I/O)_{\underline{a}\ (p+1,\underline{N}-p)}$ for some $p \in N$.

[proof] See (6-C.9) in Appendix 6.7.

(6.24) Theorem
There exists a natural partial realization of a given partial time invariant input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ if and only if the minimal partial realizations of $\underline{a}$ are unique modulo isomorphism.

[proof] See (6-C.11) in Appendix 6.7.

(6.25) Theorem
Let a partial time-invariant input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ satisfy the condition of Theorem (6.23), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ that realizes $\underline{a}$ can be obtained by the following algorithm.
Set $n :=$ rank $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$, where $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ is the finite Input/Output Matrix of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$.
1) Select the linearly independent vectors $\{S_l^{I_i}(\chi(u_{J_i}))\}$ from $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ in order of the numerical value.

2) Let the state space be K^n. Let the map $g_s : U \to K^n$ be $g_s(u_{J_i}) := \mathbf{e_i}$ for $u_{J_i} \in U$ such that $I_i = 0$. And let $g_s(u_j) := \sum_{i=1}^{j} \alpha_i \mathbf{e_i}$ for $u_j \in U$ and $\chi(u_j) := \sum_{i=1}^{j} \alpha_i \chi(u_j)$ such that $(0, u_{J_i}) < (0, u_j) < (0, u_{J_{i+1}})(or(1, u_{J_{i+1}}))$.
3) Let the output map $h_s = [a(u_{J_1}^{I_1+1}) - a(u_{J_1}^{I_1}), a(u_{J_2}^{I_2+1}) - a(u_{J_2}^{I_2}), \cdots,$
$a(u_{J_n}^{I_n+1}) - a(u_{J_n}^{I_n})]$
4) Let f_j in $F_s := [f_1 f_2, \cdots, f_n]$ be $f_j := [f_{i,1} f_{i,2}, \cdots, f_{i,n}]^T$.
Where $\underline{S_l} \underline{S_l}^{I_i} \underline{GI}(u_{J_i}) = \sum_{i=1}^{j} f_{i,j} SlIj \underline{S_l}^{I_i} \underline{GI}(u_{J_j})$, $f_{i,j} \in K$ in the sense of $F(N_{\underline{N}-p}, Y)$ and $\underline{S_l} : F(N_p, Y) \to F(N_{p-1}, Y); a \mapsto \underline{S_l}\underline{a}[; t \mapsto \underline{a}(t+1)$ for some $p \in N$.

[proof] See (6-C.12) in Appendix 6.7.

6.5 Real-Time Partial Realization Theory of Pseudo Linear System

In general, it is known that non-linear systems can only be determined by multi-experiments. In fact, in Chapter 3, it is given a condition for a general unknown black-box to be determined with a single-experiment. This condition may be very hard for us to find. However, we can look for special single-experiments to pretend multi-experiments for any Pseudo Linear System. In this section, on the results of partial realization theory in section 6.4, we will discuss single-experiment for Pseudo Linear Systems.

(6.26) Real-time partial realization problem
Let a physical object (equivalently, $a \in F(\Omega, Y)$) be a finite dimensional Pseudo Linear System. Then for given finite data $\{\underline{a}(\underline{\omega})$; an input $\underline{\omega}$ is finite length $\}$, find a Pseudo Linear System $\sigma = ((K^n, F), g, h, h^0)$ and an input $\underline{\omega}$ such that $a_\sigma(\omega) = a(\omega)$ for any $\omega \in \Omega$.

(6.27) Definition
For a finite dimensional Pseudo Linear System, if there exists a solution of real time partial realization problem, then an input $\underline{\omega} \in \Omega$ of the solution is said to be a (real time partial) realization signal.

(6.28) Lemma
Let a given time invariant input response map $a \in F(\Omega, Y)$ have the behavior of a Pseudo Linear System whose state space is less than L dimensional. Then there exists an input of finite length $\underline{\omega} \in \Omega$ such that

the following algorithm provides a finite Input/Output Matrix. Where $p := max\{L_1, L_2, \cdots, L_m\}$.
1) Find an integer L_1 such that row vectors $\{\underline{S_l}^i(\chi(u_1)) \in K^{L-1}; 0 \leq i \leq L_1 - 1\}$ are linearly independent and $\{\underline{S_l}^i(\chi(u_1)) \in K^{L-1}; 0 \leq i \leq L_1\}$ are linearly dependent. Namely, feed an input $\omega_1 := u_1^{L_1+L}$ into the plant.
2) Find an integer L_2 such that row vectors $\{\underline{S_l}^i(\chi(u_j)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq 2\}$ are linearly independent and $\{\underline{S_l}^i(\chi(uj)) \in K^{L-1}$, $\underline{S_l}^{L_2}(\chi(u_2)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq 2\}$ are linearly dependent. Namely, feed a further input $\omega_2 := u_1^{L_1+L-1}|u_2$ into the plant.
3) Find an integer L_3 such that row vectors $\{\underline{S_l}^i(\chi(u_j)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq 3\}$ are linearly independent and $\{\underline{S_l}^i(\chi(u_j)) \in K\ L-1$, $\underline{S_l}^{L_3}(\chi(u_3)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq 3\}$ are linearly dependent. Namely, feed a further input $\omega_3 := u_1^{L_3+L-1}|u_3$ into the plant.

$\vdots$

$\vdots$

m) Find an integer L_m such that row vectors $\{\underline{S_l}^i(\chi(u_j)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq m\}$ are linearly independent and $\{\underline{S_l}^i(\chi(u_j)) \in K^{L-1}$, $\underline{S_l}^{L_m}(\chi(u_m)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq m\}$ are linearly dependent. Namely, feed a further input $\omega_m := u_1^{Lm+L-1}|u_m$ into the plant.
Let $\omega = \omega_m|\omega_{m-1}| \cdots |\omega_2|\omega_1$.
Making the row vectors of a matrix from the row vectors $\{\underline{S_l}^i(\chi(uj)) \in K^{L-1}; 0 \leq i \leq L_j - 1, 1 \leq j \leq m\}$ obtained by the above iterations, we will obtain a finite Input/Output Matrix $H_{\underline{a}\ (L-1,p)}$.

[proof] See (6-D.1) in Appendix 6.7.

(6.29) Theorem
Let a given time-invariant input response map $a \in F(\Omega, Y)$ have the behavior of a Pseudo Linear System whose state space is less than L dimensional. Then there exists a realization signal such that the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, h^0)$ that realizes a can be obtained by the following algorithm:
1) Find a finite Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ upon the algorithm given in Lemma (6.28).
2) Apply the algorithm given in Theorem (6.25) to the above finite Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$.

[proof] This is obvious from the above Lemma (6.28).

6.6 Historical Notes and Concluding Remarks

We introduced Pseudo Linear Systems that are a subclass of Affine Dynamical Systems, which are close to linear systems. As an example of them, there exists a circuit with FET transistor, which is a nonlinear circuit. But we cannot find such dynamical systems, for example in Brockett [1976b], Kaasshocek, Schuppen & Ran [1990], Jakubczyk [1980] and Monaco & Cyrot [1987].

We showed that any time-invariant input response map (equivalently, any input/output map with causality and time-invariance) can be completely characterized by a modified impulse response, which may be a revised version of an impulse response in linear systems. We also showed that any Pseudo Linear Systems can be characterized by time-invariant input response maps. The set $A(N \times U, K)$ in Example (6.5) is new. Therefore, nerode equivalence for $A(N \times U, K)$ in Theorem (6.6) is new. See Section 3.4 in Chapter 3 for the comments of nerode equivalence.

It is shown that the uniqueness Theorem (6.8) holds in the sense of Pseudo Linear Systems, namely the theorem is stronger than in the sense of Affine Dynamical Systems.

Theorem (6.12) is one for finite dimensional Pseudo Linear Systems to be canonical. It can be easily understood that this theorem is an extension of theorem for finite dimensional linear systems to be canonical. We gave the quasi-reachable standard system and observable standard system that correspond to companion forms of linear systems. We gave a criterion for the behavior of finite dimensional Pseudo Linear Systems. The condition was given by finite rank of Input/Output Matrix, which is an extension of finite rank of Hankel matrix in linear systems. We also gave a realization procedure to obtain the quasi-reachable standard system from a given time-invariant input response map.

Moreover, we obtained partial Realization Theorem (6.23) and (6.24). Also we obtained a partial realization algorithm in Theorem (6.25). We could also discuss the real-time partial realization problem without any restriction. It depends upon time-invariance of input/output map.

6.7 Appendix

This Appendix is prepared for the proof of the results about Pseudo Linear Systems.

6.7.A Realization Theory

In this section, we will prove the main Theorem (6.8). To prove it, we equivalently convert the Pseudo Linear Systems to sophisticated Pseudo Linear Systems owing to results obtained in Appendix 6.7.A.1 to 6.7.A.4. In Appendix 6.7.A.5, we prove the realization theorem in the sophisticated Pseudo Linear Systems. This implies that Theorem (6.8) is proved.

6.7.A.1 Deriving Pseudo Linear Systems from Affine Dynamical Systems

We will show how Pseudo Linear Systems can be derived from Affine Dynamical Systems discussed in Chapter 5.

(6-A.1) Lemma
If the behavior a_σ of a distinguishable Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ satisfies the following time-invariant condition, then $F(u) = F(u')$ holds for any $u, u' \in U$.
Time-invariant condition :
$a_\sigma(\omega_1|\omega) - a_\sigma(\omega_1) = a_\sigma(\omega_2|\omega) - a_\sigma(\omega_2)$ for any ω, ω_1 and $\omega_2 \in \Omega$ such that $|\omega_1| = |\omega_2|$.

[proof] This is obtained by direct calculation.

Remark: An Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ represents the following equations:

$$\begin{cases} x(t+1) & = & F(\omega(t+1))x(t) + g(\omega(t+1)) \\ x(0) & = & 0 \\ \gamma(t) & = & h^0 + hx(t) \end{cases}$$

Where X is a linear space over the field K, $x(t) \in X$, $F(\omega(t))$ is a linear operator on X and $\omega(t) \in U$ for any $t \in N$. And g is a map : $U \mapsto X$, h is a linear operator : $X \to Y$ and $h^0 \in Y$.

(6-A.2) Definition
An Affine Dynamical System $\sigma = ((X, F), g, h, h^0)$ given by Lemma (6-A.1) is said to be a Pseudo Linear System. Conveniently, F in σ may be written by $F := F(u)$ for any $u \in U$. Therefore, a Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ represents the following equations:

$$\begin{cases} x(t+1) & = & Fx(t) + g(\omega(t+1)) \\ x(0) & = & 0 \\ \gamma(t) & = & h^0 + hx(t) \end{cases}$$

Where X is a linear space over the field K, F is a linear operator on X and $\omega(t) \in U$ for any $t \in N$. And g is a function : $U \mapsto X$, h is a linear operator : $X \rightarrow Y$ and $h^0 \in Y$.

Remark: This definition for a Pseudo Linear System is the same as Definition (6.4).

6.7.A.2 Linear State Structure: Free Motions

(6-A.3) Definition
A system given by the following equation is written as a pair (X, F) and it is said to be a free motion.
$x(t+1) = Fx(t)$
Where X is a linear space over the field K and a linear map $F : X \rightarrow X$.
Let (X_1, F_1) and (X_2, F_2) be free motions, then a linear map $T : X_1 \rightarrow X_2$ is said to be a free motion morphism $(X_1, F_1) \rightarrow ((X_2, F_2)$ if T satisfies $TF_1 = F_2T$.

(6-A.4) Example
Let $A(N \times U, K)$ and S_r be the same as that considered in Example (6.5). Then $(A(N \times U, K), S_r)$ is a free motion.

(6-A.5) Example
In the set $F(N, Y)$ of any map from N to Y, let S_l be the same as in Example (6.5). Then $(F(N, Y), S_l)$ is a free motion.

(6-A.6) Example

Let $X = K^n$, and $F \in K^{n\times n}$, (K^n, F) is a free motion. And it represents a state difference equation $x(t+1) = Fx(t)$ in K^n.

(6-A.7) Definition

For free motions $(A(N \times U, K), S_r)$ and $(F(N, Y), S_l)$ considered in Examples (6-A.4) and (6-A.5), a free motion morphism $A : ((A(N \times U, K), S_r) \to (F(N, Y), S_l)$ is said to be a linear input/output map. For a free motion (X, F), a free motion morphism $G : (A(N \times U, K), S_r) \to (X, F)$ is said to be a linear input map, and a free motion morphism $H : (X, F) \to (F(N, Y), S_l)$ is said to be a linear observation map.

Remark: Note that $AS_r = S_l A$ is equal to $AS_r^t = S_l^t A$ for any $t \in N$. Therefore, a linear input/output map $A : ((A(N \times U, K), S_r) \to (F(N, Y), S_l)$ represents a time-invariance of input/output maps.

As the state structure of Affine Dynamical Systems, we have introduced linear U-actions and have commented the linear Ω-modules in Appendix 5.4. We have clarified the relation between linear U-actions and linear Ω-modules. Here, we will newly introduce N-modules. Then we will relate the connection between free motions and N-modules.

(6-A.8) Definition

Let X be a linear space over the field K and a map $\phi : N \to L(X)$ be a monoid morphism, i.e., $\phi(0) = I$ (the identity map on X), and $\phi(t+s) = \phi(t)\phi(s)$ hold for any $t, s \in N$. Then a pair (X, ϕ) is said to be a N-module.

Remark: Let (X, ϕ) be an N-module. For $s \in N$, a linear map $\phi(s) : X \to X$ transfer a state $x(t)$ at time t to a state $x(t+s)$ after s time and it is linear transformation. Therefore, it represents a linear state transition equation $x(t+s) = \phi(s)x(t)$.

(6-A.9) Proposition

Let X be a linear space over the field K. For any $F \in L(X)$, a map $\phi : N \to L(X)$ obtained by the following formula $*)$ is a monoid morphism. Moreover, this correspondence is bijective.

$*) : \phi(s) = F^s$ for any $s \in N$ $\phi(0) = I$

Therefore, a free motion (X, F) uniquely corresponds to an N-module (X, ϕ) by the formula $*$).

(6-A.10) Example
For the free motion $(A(N \times U, K), S_r)$ considered in Example (6-A.4), N-module $(A(N \times U, K), S_r)$ corresponding to it is given by setting $S_r(t) := S_r^t$ for any $t \in N$.

(6-A.11) Example
For the free motion $(F(N, Y), S_l)$ considered in Example (6-A.5), the N-module corresponding to it is given by $(F(N, Y), S_l)$, where $S_l(s) : F(N, Y) \to F(N, Y); \gamma \mapsto S_l(s)\gamma[; t \mapsto \gamma(t + s)]$.

Here, we have introduced N-module and discussed a connection between free motions and N-modules.

(6-A.12) Definition
Let $(X_1, F_1)[(X_1, \phi_1)]$ and $(X_2, F_2)[(X_2, \phi_2)]$ be free motions [N-module], then a linear map $T : X_1 \to X_2$ is said to be a free motion morphism :$(X_1, F_1) \to (X_2, F_2)$ [an N-module morphism : $(X_1, \phi_1) \to (X_2, \phi_2)$] if T satisfies $TF_1 = F_2T$ [$T\phi_1(t) = \phi_2(t)T$ for any $t \in N$].

(6-A.13) Proposition
Let i=1 and 2. Let (X_i, F_i) be a free motion, (X_i, ϕ_i) be an N-module corresponding to it. Then the following two conditions are equivalent:
1) A morphism T is a free motion morphism : $(X_1, F_1) \to (X_2, F_2)$.
2) A morphism T is an N-module morphism : $(X_1, \phi_1) \to (X_2, \phi_2)$.

[proof] It is obvious that condition 1) is equivalent to condition 2).

In apprndix 5.4 we have introduced the sub U-action, the quotient U-action and the product U-action. Here we will introduce sub free motions, quotient free motions and product free motions. We can introduce sub N-modules, quotient N-modules and product N-modules in the same way.

(6-A.14) Sub free motions
Let (X, F) be a free motion and $Y \subseteq X$ be invariant sub-space under F, i.e., $Fy \in Y$ for any $y \in Y$. Let $F_Y := F|_Y$ (restriction of the map F to Y), then (Y, F_Y) is a free motion, and it is said to be a sub free motion of (X, F).

(6-A.15) Quotient free motions
Let (X, F) be a free motion and a linear equivalence relation R in X be consistent with F. Namely, an equivalence relation R is given by $x_1Rx_2 \iff x_1 - x_2 \in S$ for some linear sub space $S \subset X$, and x_1Rx_2 implies Fx_1RFx_2. Then we can consider a quotient linear space $X/_R = X/S$. Therefore, we can obtain a quotient free motion $(X/S, \tilde{F})$.
Where $\tilde{F} : X/S \to X/S; [x] \mapsto [Fx]$.

(6-A.16) Corollary
Any free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ can be normally decomposed into $X_1 \overset{\pi}{\to} X_1/\ker T \overset{T^b}{\to} \text{im } T \overset{j}{\to} X_2$, where π is the canonical surjection, T^b is the isomorphism associated with T, j is the canonical injection and they are free motion morphisms respectively.

(6-A.17) Product free motions
Let (X_1, F_1) and (X_2, F_2) be free motions and define $(F_1 \times F_2) : X_1 \times X_2 \to X_1 \times X_2; (x_1, x_2) \mapsto (F_1x_1, F_2x_2)$ for the product space $X_1 \times X_2$. Then, $(X_1 \times X_2, F_1 \times F_2)$ is a free motion, and it is said to be a product free motion of (X_1, F_1) and (X_2, F_2).

(6-A.18) Proposition
$A(N \times U, K)^* = TF(\Omega, Y)$.
Where $A(N \times U, K)^*$ is a set of any linear maps from $A(N \times U, K)$ to Y, and $TF(\Omega, Y) := \{a \in F(\Omega, Y); a \text{ is a time-invariant response map }\}$.

[proof] For any $a \in TF(\Omega, Y)$, set $\tilde{} : a \mapsto \tilde{a}[; \sum_{n,u} \lambda(n, u)\mathbf{e_{(n,u)}} \mapsto \sum_{(n,u)} \lambda(n, u)(a(u^{n+1}) - a(u^n))$, then $\tilde{a} \in A(N \times U, K)^*$ holds. For any $\tilde{a} \in A(N \times U, K)^*$, set $e* : \tilde{a} \mapsto \tilde{a} \cdot e[; \omega \mapsto \tilde{a}(\mathbf{e}_{(\mathbf{n}, \omega(1))})]$, then $\tilde{a} \cdot e \in TF(\Omega, Y)$ holds. Here, $e * \cdot \tilde{} = I$ and $\tilde{} \cdot e* = I$ hold. Therefore $TF(\Omega, Y)$ is a concrete expression of $A(N \times U, K)^*$, we obtain $A(N \times U, K)^* * = TF(\Omega, Y)$.

6.7.A.3 Free Motions with an Affine Map

In Appendix 5.4, we have introduced linear U-actions with an affine map and linear U-actions with an affine input map, and have shown that they are equivalent. In this section, we will introduce free motions with an affine map and free motions with an affine input map, and will show that they are equivalent. Moreover, we discuss quasi-reachability of free motions with an affine map.

(6-A.19) Definition

For a free motion (X, F) and a map $g : U \to X$, a collection $((X, F), g)$ is said to be a free motion with an affine map.
A free motion with an affine map $((X, F), g)$ represents the following equations.
$x(t + 1) = Fx(t) + g(\omega(t + 1))$
Where $x(t) \in X$, $\omega(t) \in U$.
For the reachable set $\{\sum_{j=1} |\omega|F^{|\omega|-j}g(\omega(j)); \omega \in \Omega\}$, the smallest linear space that contains it is equal to X, then $((X, F), g)$ is said to be quasi-reachable.

(6-A.20) Example

For the free motion $(A(N \times U, K), S_r)$ considered in Example (6-A.10) and the map $\eta : U \to A(N \times U, K); u \mapsto \mathbf{e_{(0,u)}}$, $(A(N \times U, K), S_r), \eta)$ is a free motion with an affine map and quasi-reachable.

(6-A.21) Example

For the free motion $(F(N, Y), S_l)$ considered in Example (6-A.11) and a map $\chi : U \to F(N, Y); u \mapsto \chi(u)[; t \mapsto (\chi(u))(t) = a(\omega|u) - a(\omega)]$ for time-invariant input response map $a \in F(\Omega, Y)$ and $|\omega| = t$, $\omega \in \Omega$.
$((F(N, Y), S_l), \chi)$ is a free motion with an affine map.

(6-A.22) Definition

For free motions with an affine map $((X_1, F_1), g_1)$ and $((X_2, F_2), g_2)$, a free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $Tg_1 = g_2$ is said to be a free motion morphism with an affine map $T : ((X_1, F_1), g_1) \to ((X_2, F_2), g_2)$.

(6-A.23) Proposition

For any free motion with an affine map $((X, F), g)$, there exists a unique free motion morphism with an affine map $G : (A(N \times U, K), S_r), \eta) \to ((X, F), g)$, where $G(\mathbf{e_{(0,u)}}) = g(u)$ for any $u \in U$.
Conversely, for any free motion morphism $G : (A(N \times U, K), S_r) \to (X, F)$, $((X, F), g)$ given by $g(u) = G(\mathbf{e_{(0,u)}})$ is a free motion with an affine map.

[proof] Let $G(\mathbf{e_{(0,u)}}) = g(u)$. Since $GS_r = FG$, G can be extend to $\{\mathbf{e_{(t,u)}}; t \in N, u \in U\}$. $\{\mathbf{e_{(t,u)}}; t \in N, u \in U\}$ being the basis in $A(N \times U, K)$, G is unique. The latter part is obvious.

Remark 1: According to Proposition (6-A.23), a linear input map $G : (A(N\times U, K), S_r) \to (X, F)$ corresponds to an affine map $g : U \to X$ is determined uniquely and this correspondence is isomorphism.
Remark 2: If a free motion with an affine map $((X, F), g)$ in Proposition (6-A.23) is replaced with $((F(N, Y), S_l), \chi)$ considered in Example (6-A.21), then a linear input map $A : (A(N\times U, K), S_r) \to ((F(N, Y), S_l), \chi)$ uniquely corresponds to a time-invariant input response map $a \in F(\Omega, Y)$, and this correspondence is isomorphism.

(6-A.24) Corollary
For any linear input/output map $A : (A(N \times U, K), S_r) \to ((F(N, Y), S_l)$, there uniquely exists a time-invariant input response map $a \in F(\Omega, Y)$ such that $a(\omega|u) - a(\omega) = A(\mathbf{e}_{(\mathbf{0},\mathbf{u})})(t)$ for any $t \in N$, $\omega \in \Omega$ and $|\omega| = t$. This correspondence is isomorphism.

[proof] If a free motion with an affine map $((X, F), g)$ in Proposition (6-A.23) is replaced with $((F(N, Y), S_l), \chi)$, this corollary is obtained.

Remark: A linear input/output map $A : (A(N\times U, K), S_r) \to ((F(N, Y), S_l)$ satisfies the following equation by Definition (6-A.7):
$AS_r = S_l A$ or equivalently, $AS_r^t = S_l^t A$ for any $t \in N$. This equation means that a linear input/output map A satisfies time-invariance.
By Corollary (6-A.24), an input response map $a \in F(\Omega, Y)$ corresponding to A may be said to be a time-invariant input response map.

(6-A.25) Proposition
A free motion with an affine map $((X, F), g)$ is quasi-reachable if and only if the corresponding linear input map G is surjective.

[proof] The quasi-reachability of $((X, F), g)$ means that the liner hull of the reachable set $\{\sum_{j=1} |\omega| F^{|\omega|-j} g(\omega(j)); \omega \in \Omega\}$ is equal to X. On the other hand, the surjection of G means that $\ll \{G(\lambda) = \sum_{(n,u)} F^n g(u); \lambda = \sum_{(n,u)} \mathbf{e}_{(\mathbf{n},\mathbf{u})}\} \gg$ is equal to X. Therefore, this proposition is obtained.

6.7.A.4 Free Motions with a Readout Map

In this section, we introduce free motions with a readout map and free motions with a linear observation map, and show that they are equivalent. Moreover, we discuss distinguishablity of free motions with a readout map.

(6-A.26) Definition
For a free motion (X, F) and a linear map $h : X \to Y$, a collection $((X, F), h)$ is said to be a free motion with a readout map. A free motion with a readout map $((X, F), h)$ represents the following equations:

$$\begin{cases} x(t+1) & = & Fx(t) \\ \gamma(t) & = & hx(t) \end{cases}$$

Where X is a linear space over the field K, F is a linear operator on X, $x(t) \in X$. And h is a linear operator : $X \to Y$
For any $n \in N$, if $hF^n x_1 = hF^n x_2$ implies $x_1 = x_2$, then $((X, F), h)$ is called observable.
Let $((X_1, F_1), h_1)$ and $((X_2, F_2), h_2)$ be free motions with a readout map, then a free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $h_1 = h_2 T$ is said to be a free motion morphism with a readout map $T : ((X_1, F_1), h_1)$
$\to ((X_2, F_2), h_2)$.

(6-A.27) Example
For the free motion $(A(N \times U, K), S_r)$ considered in (6-A.4) and any time-invariant input response map $a \in F(\Omega, Y)$, $((A(N \times U, K), S_r), \bar{a})$ is free motion with a readout map. Where a linear map $\bar{a} : A(N \times U, K) \to Y$ is given by $\bar{a}(\mathbf{e}_{(\mathbf{n},\mathbf{u})}) = a(u^{n+1}) - a(u^n)$ for any $u \in U$.

(6-A.28) Example
Regarding the free motion $((F(N, Y), S_l)$ in Example (6-A.5), by defining a linear map $0 : (F(N, Y) \to Y; \gamma \mapsto \gamma(0)$, $((F(N, Y), S_l), 0)$ is a free motion with a readout map and it is observable.

(6-A.29) Proposition
For any free motion with a readout map $((X, F), h)$, there exists a unique linear observation map $H : (X, F) \to ((F(N, Y), S_l)$ which satisfies $h = 0 \cdot H$, where $(Hx)(t) = hF^t x$ holds for $x \in X, t \in N$.

[proof] Let $((X, F), h)$ be any. Defining $(Hx)(t) := hF^t x$ (for any $x \in X, t \in N$), we can obtain a linear observation map $H : (X, F) \to ((F(N, Y), S_l)$ and H satisfies $h = 0 \cdot H$. Next, we will show uniqueness. Let H be a linear observation map : $(X, F) \to ((F(N, Y), S_l)$ which satisfies $h = 0 \cdot H$, then $(Hx)(t) = (S_l(t)Hx)(0) = 0(S_l(t)Hx) = 0(HF^t x) = hF^t x$ hold for any $x \in X, t \in N$. Therefore, H is unique.

Remark 1: According to Proposition (6-A.29), a linear observation map $H : (X, F) \to ((F(N, Y), S_l)$ corresponds uniquely to a linear map $h : X \to Y$ and this correspondence is isomorphism.
Remark 2: If $((X, F), h)$ in Proposition (6-A.29) is replaced with $(((F(N, Y), S_l), 0)$ considered in (6-A.28), a linear observation map : $(A(N \times U, K), S_r) \to ((F(N, Y), S_l)$ is a linear input/output map.

A following proposition is obtained easily by noticing the definition of observability and Proposition (6-A.29).

(6-A.30) Proposition
A free motion with readout map $((X, F), h)$ is observable if and only if the corresponding linear observation map $H : (X, F) \to ((F(N, Y), S_l)$ is injective.

6.7.A.5 Pseudo Linear Systems

In this section, we introduce sophisticated Pseudo Linear Systems, and show that Pseudo Linear Systems (said to be a naive pseudo linear) introduced in Definition (6.4) and sophisticated Pseudo Linear Systems are considered as the same thing.

(6-A.31) Definition
A collection $\Sigma = ((X, F), G, H, h^0)$ is said to be a sophisticated Pseudo Linear System, if G is a linear input map : $((A(N \times U, K), S_r) \to (X, F)$ and H is a linear observation map : $(X, F) \to (F(N, Y), S_l)$.
A linear input/output map $A_\Sigma = H \cdot G : (A(N \times U, K), S_r) \to (F(N, Y), S_l)$ is said to be a behavior of Σ.
For a linear input/output map A and some $a(1) \in Y$, if $A_\Sigma = A$ and $h^0 = a(1)$, then sophisticated Pseudo Linear System Σ is called a realization of $(A, a(1))$.
A sophisticated Pseudo Linear System $\Sigma = ((X, F), G, H, h^0)$ is said to be canonical if G is surjective and H is injective.

For $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$, a free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $TG_1 = G_2$ and $H_1 = H_2 T$ is said to be a sophisticated Pseudo Linear System morphism $: \Sigma_1 \to \Sigma_2$.
If T is surjective and injective then $T : \Sigma_1 \to \Sigma_2$ is said to be an isomorphism.

(6-A.32) Example
Regarding the free motion $(A(N \times U, K), S_r)$ in Example (6-A.4), identity map I on $A(N \times U, K)$ and a linear input/output map $A : (A(N \times U, K), S_r) \to ((F(N, Y), S_l)$, a collection $((A(N \times U, K), S_r), I, A, a(\mathit{1}))$ is a sophisticated Pseudo Linear System with the behavior $(A, a(\mathit{1}))$.

Regarding the free motion $((F(N, Y), S_l)$ in Example (6-A.5), a linear input/output map A and identity map I on $F(N, Y)$, then a collection $(((F(N, Y), S_l), A, I, a(\mathit{1}))$ is a sophisticated Pseudo Linear System with the behavior $(A, a(\mathit{1}))$.

In this situation, we consider the relation between sophisticated Pseudo Linear Systems and naive ones.

(6-A.33) Proposition
For any sophisticated Pseudo Linear System $\Sigma = ((X, F), G, H, h^0)$, there exists a unique naive Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ corresponding to the sophisticated Pseudo Linear System Σ by two equations (a.1) and (a.2).
$G(\omega) = \sum_{j=1}^{|\omega|} F^{|\omega|-j} g(\omega(j))$ for $\omega \in \Omega$ .... (a.1)
$Hx(t) = hF^t x$ for any $x \in X$ and $t \in N$ .... (a.2)
This correspondence is isomorphic in the category's sense.

[proof] It is easily obtained from Remark 1 of Proposition (6-A.23) and Remark 1 of Proposition (6-A.29).

6.7.A.6 Sophisticated Pseudo Linear Systems

In this section, we will prove Realization Theorem (6.8). According to the Remark 2 in Proposition (6-A.23) (or the Remark 2 in Proposition (6-A.29)) and Proposition (6-A.33), the realization theorem can be replaced with the following Theorem (6-A.34). Therefore, proving this theorem implies proving Realization Theorem (6.8).

(6-A.34) (Sophisticated) Realization Theorem
For any linear input/output map $A : (A(N \times U, K), S_r) \to ((F(N, Y), S_l)$ and some $a(1) \in Y$, there exist at least two sophisticated canonical Pseudo Linear Systems that realize $(A, a(1))$ (existence part).
Let $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$ be sophisticated canonical Pseudo Linear Systems that have the same behavior, then there exists an isomorphism $T : \Sigma_1 \to \Sigma_2$ (uniqueness part).

[proof] The next Corollary (6-A.35) signifies proving the existence part. Moreover, Remark in Corollary (6-A.39) signifies proving the uniqueness.

(6-A.35) Corollary
For any linear input/output map $A : (A(N \times U, K), S_r) \to ((F(N, Y), S_l)$ and some $a(1) \in Y$, the following sophisticated Pseudo Linear Systems (1) and (2) are canonical realizations of $(A, a(1))$.
(1) $\Sigma_q = ((A(N \times U, K)/\text{ker } A, \tilde{S_r}), \pi, A^i, a(1))$.
Where π is the canonical surjection : $A(N \times U, K) \to A(N \times U, K)/\text{ker } A$ and A^i is given by $A^i = j \cdot A^b$ for $A^b : A(N \times U, K)/\text{ker } A \to \text{im } A$ being isomorphic with A and j being the canonical injection : $\text{im } A \to F(N, Y)$.
(2) $\Sigma_s = ((\text{im } A, S_l), A^s, j, a(1))$.
Where $A^s = A^b \cdot j$.

[proof] This can be obtained easily by Corollary (6-A.16), Example (6-A.32), the definition of canonicality and the behavior.

Next, to prove the uniqueness part of Theorem (6-A.33), we introduce the following morphism $Mor(\Sigma_1, \Sigma_2)$ from a sophisticated Pseudo Linear System Σ_1 to another sophisticated Pseudo Linear System Σ_2.
Where Σ_1 and Σ_2 are given by $\Sigma_1 = ((X_1, F_1), G_1, H_1, h0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$ respectively.
$Mor(\Sigma_1, \Sigma_2) := \{$ a relation $T : X_1 \to X_2; GrT_{12}^{min} \subseteq GrT_{12} \subseteq GrT_{12}^{max}\}$.
Where GrT_{12}^{min}, GrT_{12} and GrT_{12}^{max} denote the graph of $T_{12}^{min} := G_2 \cdot G_1^{-1}, T_{12}$ and $H_2^{-1} \cdot H_1$ respectively.
Why this morphism is introduced depends on the next lemma.

(6-A.36) Lemma
$A_{\Sigma_1} = A_{\Sigma_2}$ if and only if $Mor(\Sigma_1, \Sigma_2) \neq \emptyset$.

[proof] This can be proved the same as in Matsuo [1981].

(6-A.37) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold.
(1) If G_1 of Σ_1 is surjective, then dom $T_{12}^{min} = X_1$ holds, where dom T_{12}^{min} denotes the domain of T_{12}^{min}.
(2) If H_2 of Σ_2 is injective, then T_{12}^{max} is a partial function : $X_1 \to X_2$.

[proof] This can be proved the same as in Matsuo [1981].

(6-A.38) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, then GrT_{12}^{max} is an invariant sub-product linear U-action of (X_1, F_1) and (X_2, F_2).

[proof] By the definition of $T_{12}^{max}, GrT_{12}^{max} = \{(x_1, x_2) \in X_1 \times X_2; H_1 x_1 = H_2 x_2\}$ holds. Let (x_1, x_2) and $(x_1', x_2') \in GrT_{12}^{max}, i.e., H_1 x_1 = H_2 x_2$ and $H_1 x_1' = H_2 x_2'$ hold. $H_1(x_1 + x_1') = H_1 x_1 + H_1 x_1' = H_2 x_2 + H_2 x_2' = H_2(x_2 + x_2')$ hold. This implies $(x_1 + x_1', x_2 + x_2') \in GrT_{12}^{max}$. For $k \in K$ and $(x_1, x_2) \in GrT_{12}^{max}$, $(kx_1, kx_2) \in GrT_{12}^{max}$ holds. Moreover, for any $(x_1, x_2) \in GrT_{12}^{max}$, $H_1 F_1 x_1 = S_l H_1 x_1 = S_l H_2 x_2 = H_2 F_2 x_2$ hold. Therefore, we obtain $(F_1 x_1, F_2 x_2) \in GrT_{12}^{max}$. Therefore, $GrT_{12}^{max} \subseteq X_1 \times X_2$ is invariant under $F_1 \times F_2$. Therefore, $(GrT_{12}^{max}, F_1 \times F_2)$ is a free motion.

(6-A.39) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, G_1 be surjective and H_2 be injective, then $T_{12}^{min} = T_{12}^{max}$ holds and T_{12} is a Pseudo Linear System morphism : $\Sigma_1 \to \Sigma_2$ by setting $T_{12} = T_{12}^{min}$.

[proof] If G_1 is surjective and H_2 is injective, then Lemma (6-A.39) implies that $T_{12} \in Mor(\Sigma_1, \Sigma_2)$ is unique, $T_{12} \cdot G_1 = G_2$ and $H_2 T_{12} = H_1$ hold. Owing to Lemma (6-A.38), T_{12} is a free motion morphism : $(X_1, F_1) \to (X_2, F_2)$.

Remark: The uniqueness part of (sophisticated) Realization Theorem (6-A.34) for time-invariant input response maps is proven by sophisticated Pseudo Linear Systems being canonical and Lemma (6-A.39).

6.7.B Finite Dimensionality

In this Appendix, we will give proofs for theorems, propositions and corollaries stated in section 6.3.

6.7.B.1 Finite Dimensional Free Motions with an Affine Map

In Appendix 6.7.A, the free motions were introduced. In this section, we consider those whose state space is finite dimensional. Then it is shown that finite dimensional free motions can be represented by matrix expressions.

(6-B.1) Definition
A free motion (X, F) whose X is finite (n) dimensional is said to be a finite dimensional (n dimensional) free motion.

In Appendix 6.7.A.3, we showed that an initial object of any free motion with an affine map $((X, F), g)$ is $(A(N \times U, K), S_r), \eta)$ and the quasi-reachability of $((X, F), g)$ implies a surjection of the corresponding linear input map G. In this section, we will give a criterion for being quasi-reachable of finite dimensional free motions with an affine map. Introducing the quasi-reachable standard form, we show that it is a representative of free motion with an affine map.

Let $((X, F), g)$ be a free motion with an affine map and G be the linear input map corresponding to an affine map g, namely, a free motion morphism $G : (A(N \times U, K), S_r) \to (X, F)$ which satisfies $G(\mathbf{e}_{(\mathbf{0},\mathbf{u})}) = g(u)$ for any $u \in U$.

Let $LR(i)$ be the linear hull of reachable set by input whose length is within i, i.e., $LR(i) :=\ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j} g(\omega(j)); \omega \in \Omega_{i+1}\} \gg$.
Where $\Omega_i := \{\omega \in \Omega; |\omega| \leq i\}$.
Then the following formula holds.
$LR(0) =\ll \{g(u); u \in U\} \gg$,
$LR(i+1) = LR(i)+ \ll \{Fx + g(u); u \in U, x \in LR(i)\} \gg$.
Therefore, the following sequence can be obtained.
$LR(0) \subseteq LR(1) \subseteq \cdots \subseteq LR(i) \subseteq \cdots \subseteq LR(\infty)$.
And $LR(n) = G(A(N \times U, K)_n)$ holds.
Where $A(N \times U, K)_n$ denotes $\{\sum_{(q,u)} \lambda(q, u)\mathbf{e}_{(\mathbf{q},\mathbf{u})} \in A(N \times U, K), q \leq n$ for $n \in N\}$.
Moreover, let $G_l := G \cdot J_l$, where J_l is the canonical injection

$: A(N \times U, K)_l \to A(N \times U, K)$. Then the above sequence can be rewritten as the following:
im $G_0 \subseteq$ im$G_1 \subseteq \cdots \subseteq$ im $G_i \subseteq \cdots \subseteq$ im G_∞.

Then we can obtain next lemma easily.

(6-B.2) Lemma
If im $G_{j-1} =$ im G_j for an integer $j \in N$, then im $G_j =$ im G_{j+1}.

[proof] By the formula, im G_j = im $G_{j-1}+ \ll \{Fx + g(u); u \in U, x \in$ im $G_{j-1}\} \gg$ holds. By assumption im G_{j-1} = im G_j, im $G_{j+1} =$ im $G_{j-1}+ \ll \{Fx + g(u); u \in U, x \in$ im $G_{j-1}\} \gg=$ im G_j holds.

(6-B.3) Lemma
For any free motion with an affine map $((K^n, F), g)$, then im $G_{n-1} =$ im G always holds. Therefore, ((im $G_{n-1}, F), g)$ is a quasi-reachable free motion with an affine map.

[proof] This is a direct consequence of Lemma (6-B.2) and the definition of quasi-reachability.

(6-B.4) Proposition
Let $((K^n, F), g)$ be a free motion with an affine map, then $((K^n, F), g)$ is quasi-reachable if and only if im$G_{n-1} = K^n$ holds.

[proof] The necessary and sufficient condition for being quasi-reachable of $((K^n, F), g)$ is that im $G = K^n$. By Lemma (6-B.3), this is equivalent to im $G_{n-1} = K^n$. Consequently, the proposition holds.

(6-B.5) Proposition
Let $((K^n, F), g)$ be a quasi-reachable free motion with an affine map, then im G_{j-1} is more than j dimensional for any integer j $(1 \leq j \leq n)$.

[proof] For any integer j, let's assume that there does not exist j linearly independent vectors in im G_{j-1}. And if im $G_{j-2} \subset$ im G_{j-1} holds, then the condition contradicts non existence of j vectors. Therefore, im $G_{j-2} =$ im $G_{j-1} = \cdots =$ im G_∞ holds and im G_∞ has no more than j vectors. This contradicts the quasi-reachability of $((K^n, F), g)$.

(6-B.6) Proposition

Let $((K^n, F), g)$ be a free motion with an affine map. $((K^n, F), g)$ is quasi-reachable if and only if
rank $[g(u_1), g(u_2), \cdots, g(u_m), Fg(u_m), Fg(u_2), \cdots, Fg(u_m), \cdots,$
$F^{n-2}g(u_m), F^{n-1}g(u_1), \cdots, F^{n-1}g(u_m)] = n$ holds.

[proof] This can be obtained by Proposition (6-B.4).

(6-B.7) Definition

Let $((K^n, F_s), g_s)$ be a quasi-reachable free motion. If $((K^n, F_s), g_s)$ satisfies the following conditions, then it is said to be the quasi-reachable standard form:
1) $\mathbf{e_i} = F_s^{I_i} g_s(u_{J_i})$ holds for a set $\{(I_i, u_{J_i}) \in N \times U,\ 1 \leq i \leq n\}$.
2) $(I_1, u_{J_1}) < (I_2, u_{J_2}) < \cdots < (I_n, u_{J_n})$ holds.
3) $I_i < i$
4) $F_s^p g_s(u_q) = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}$ holds for any $(p, u_q) \in N \times U$ such that $(I_j, u_{J_j}) < (p, u_q) < (I_{j+1}, u_{J_{j+1}})$.
Where $\alpha_i \in K$ and $\mathbf{e_i} = [0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T$.

(6-B.8) Proposition

For any quasi-reachable free motion with an affine map $((K^n, F), g)$, there uniquely exists the quasi-reachable standard form $((K^n, F_s), g_s)$ which is isomorphic to it.

[proof] We select the set of linearly n independent vectors $\{F^{I_i} g(u_{J_i}); (I_i, u_{J_i}) \in N \times U, 1 \leq i \leq n\}$ among $\{F^i g(u_j); i, j \in N\}$ in the order of numerical value of $N \times U$. Then the condition $I_i < i$ for $i(1 \leq i \leq n)$ holds by Proposition (6-B.5). We introduce a linear operator $T : K^n \to K^n$ by setting $TF^{I_i} g(u_{J_i}) = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_s := TFT^{-1}$ and $Tg(u) := g_s(u)$ for any $u \in U$, then $F \in K^{n \times n}$ and a collection $((K^n, F_s), g_s)$ is a free motion with an affine map. Since $TF^{I_i} g(u_{J_i}) = \mathbf{e_i}$ for $i(1 \leq i \leq n)$, the state $\mathbf{e_i}$ is a reachable state by a pair (I_i, J_i), $I_i < i - 1$. T is a free motion morphism with an affine map : $((K^n, F), g) \to ((K^n, F_s), g_s)$. T preserves the linear independence and dependence. Therefore, $((K^n, F_s), g_s)$ is the quasi-reachable standard form. Moreover, we can show the uniqueness of it comes from the selection of $\{(I_i, u_{J_i}) \in N \times U, 1 \leq i \leq n\}$.

Remark: There are many equivalences in the category of free motions with an affine map, and this proposition says that the equivalences can be represented as quasi-reachable standard forms.

6.7.B.2 Finite Dimensional Free Motions with a Readout Map

In Appendix 6.7.A.4, we showed that the final object of any free motions with a readout map $((X, F), h)$ is $((F(N, Y), S_l), 0)$ and the distinguishability of $((X, F), h)$ implies injection of the corresponding linear observation map H. In this section, we will give a criterion for being distinguishable of finite dimensional free motions with a readout map. Introducing the distinguishable standard form, we show that it is a representative of free motions with a readout map.

Let $((X, F), h)$ be a free motion with a readout map and H be the linear observation map corresponding to a readout map h, namely, a free motion morphism $H : (X, F) \to ((F(N, Y), S_l)$ which satisfies $0 \cdot H = h$.
Let $LO(i)$ be the linear hull of reachable set by output whose length is within i, i.e., $LO(i) := \{\sum \lambda_i x_j^* \in X^*; x_j^* = hF^j, 0 \le j \le i, \lambda_j \in K\}$. Then the following sequence holds.
$LO(0) \subseteq LO(1) \subseteq \cdots \subseteq LO(i) \subseteq \cdots \subseteq LO(\infty)$.

Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(N, Y) \to F(N_l, Y)$, $F(N_l, Y) := \{a \in F(N, Y); a : N_l \to Y\}$ and $N_l := \{j \in N; j \le l\}$.
Then ker $H_l = LO(l)^0$ holds, i.e., ker $H_l = \{x \in X; hx = 0 \text{ for } h \in LO(l)\}$.
Moreover, ker $H = LO(\infty)^0$ holds.

(6-B.9) Lemma
For any free motion with a readout map $((K^n, F), h)$, $LO(n-1) = \ll hF^N \gg$ holds.
Where $hF^N = \{hF^i; i \in N\}$.

[proof] This can be obtained the same way as Lemma (6-B.3).

(6-B.10) Proposition
For any free motion with a readout map $((K^n, F), h)$, ((ker $H_{n-1}, F)$ is a sub free motion of (K^n, F) and $((K^n/\text{ker } H_{n-1}, \bar{F}), \bar{h})$ is a distinguishable free motion with a readout map.

[proof] Let H be the corresponding linear observation map to h. By Lemma (6-B.9), $LO(n-1) = \ll hF^N \gg$ holds. Therefore, $\ker H_{n-1} = \ker H$ holds. Because H is a free motion morphism : $(K^n, F) \to (F(N,Y), S_l)$, $(\ker H_{n-1}, F)$ is a sub free motion of (K^n, F). Therefore, $((K^n/\ker H_{n-1}, \bar{F}), \bar{h})$ can be introduced, and become an observable free motion with a readout map.

(6-B.11) Proposition
Let $((K^n, F), h)$ be a free motion with a readout map. $((K^n, F), h)$ is observable if and only if $LO(n-1) = K^{p\times n}$ holds.

[proof] This can be obtained the same as Proposition (6-B.4).

(6-B.12) Proposition
If $((K^n, F), h)$ is observable, then $LO(j-1)$ is more than j dimensional for any j $(1 \leq j \leq n)$.

[proof] This can be obtained the same as Proposition (6-B.5).

(6-B.13) Proposition
Let $((K^n, F), h)$ be a free motion with a readout map. $((K^n, F), h)$ is observable if and only if rank $[h^T, (hF)^T, (hF^2)^T, \cdots, (hF^{n-1})^T] = n$ holds. Where T denotes the transpose.

[proof] This can be obtained the same as Proposition (6-B.6).

(6-B.14) Definition
Let $((K^n, F_o), h_o)$ be an observable free motion with a readout map. If $((K^n, F_o), h_o)$ satisfies the following conditions, then it is said to be the observable standard form:
1) $\mathbf{e_i}^T = h_o F_o^{i-1}$ holds for $i (1 \leq i \leq n)$.
2) $h_o F_o^n = \sum_{i=1}^n \alpha_i \mathbf{e_i}$ holds.

Remark: If $((K^n, F), h)$ is the observable standard form, note that $h = \mathbf{e_1}^T$.

(6-B.15) Propositon
For any observable free motion with a readout map $((K^n, F), h)$, there exists uniquely the observable standard form $((K^n, F_o), \mathbf{e_1}^T)$ which is isomorphic to it.

[proof] We select the set of n linearly independent vectors $\{hF^{i-1}; 1 \leq i \leq n\}$. We introduce a linear operator $T : K^n \to K^n$ by setting $hF^{i-1} = \mathbf{e_i}^T$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_o := TFT^{-1}$, then $F_o \in K^{n \times n}$ and a collection $((K^n, F_o), \mathbf{e_1}^T)$ is a free motion with readout map. T is a free motion morphism with readout : $((K^n, F), h) \to ((K^n, F_o), \mathbf{e_1}^T)$. T preserves the linear independence and dependence. Hence $((K^n, F_o), \mathbf{e_1}^T)$ is the observable standard form.

Remark: There are many equivalences in the category of free motions with a readout map, and this proposition says that the equivalences can be represented as the observable standard forms.

6.7.B.3 Finite Dimensional Pseudo Linear Systems

This section is prepared for the proofs of Representation Theorems (6.15) and (6.17) for finite dimensional canonical Pseudo Linear Systems.

(6-B.16) Proof of Representation Theorem (6.15)
Note that a free motion with an affine map in the quasi-reachable standard system is the quasi-reachable standard form.
Let $\sigma = ((K^n, F), g, h, h^0)$ be any finite dimensional canonical Pseudo Linear System. For the quasi-reachable standard form $((K^n, F_s), g_s)$ and a free motion morphism with an affine map $T : ((K^n, F), g) \to ((K^n, F_s), g_s)$ introduced in the proof of Theorem (6-B.8), let $h_s := h \cdot T^{-1}$. Then T is a Pseudo Linear System morphism : $\sigma = ((K^n, F), g, h, h^0) \to \sigma_s = ((K^n, F_s), g_s, h_s, h^0)$. T is bijective and σ_s is the only quasi-reachable standard system. By Corollary (6.9), the behaviors of σ and σ_s are the same.

(6-B.17) Proof of Representation Theorem (6.17)
Note that a linear U-action with a readout map in the observable standard system is the observable standard form.
Let $\sigma = ((K^n, F), g, h, h^0)$ be any finite dimensional canonical Pseudo Linear System. For the observable standard form $((K\ n, F_o), \mathbf{e_1}^T)$ and a free motion morphism with a readout map $T : ((K^n, F), h) \to ((K^n, F_o), \mathbf{e_1}^T)$ introduced in the proof of Proposition (6-B.15), let $g_o(u) := Tg(u)$. Then T is a Pseudo Linear System morphism : $\sigma = ((K^n, F), g, h, h^0) \to \sigma_o = ((K^n, F_o), g_o, \mathbf{e_1}^T, h^0)$. T is bijective and σ_o is obviously the only observable standard system. By Corollary (6.9), the behaviors of σ and σ_o are the same.

6.7.B.4 Existence Criterion for Pseudo Linear System

This section is prepared for the proofs of the theorem for existence criterion (6.19). Let $G_l = G \cdot J_l$, where J_l is the canonical injection : $A(N \times U, K)_l \to A(N \times U, K)$. Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(N, Y) \to F(N_l, Y)$.

(6-B.18) Proof of Theorem (6.19)
Let A be the linear input/output map corresponding to a time-invariant input response map $a \in F(\Omega, Y)$. Then $A(\mathbf{e_{(s,u)}})(t) = a(u^{s+t+1}) - a(u^{s+t})$ holds. Therefore, a can be represented by the infinite matrix $(I/O)_a$ considered in Definition (6.18).
By setting $A_{(l,m)} := P_m \cdot A \cdot J_l$, then $A_{(l,m)}$ can be represented by a partial Input/Output Matrix $(I/O)_{a\ (l,m)}$ of the Input/Output Matrix $(I/O)_{\underline{a}}$ considered in Definition (6.22).
Where $(I/O)_{a\ (l,m)} = [a(u^{s+t+1}) - a(u^{s+t})]$ for $t \leq l$ and $s \leq m$.
First, we show 1) $\Longrightarrow$ 2). By Theorem (6.6) and Corollary (6-A.35), im A is n dimensional. If im $A_{n-1} \neq$ im A_n then the dimension of im A_n is $n+1$ or more by Lemma (6-B.5), hence im $A_{n-1} =$ im $A_n = \cdots =$ im A hold. Consequently, there exist n linearly independent vectors in $\{S_l^i(\chi(u)); u \in U, i \in N, 1 \leq i \leq n\}$, but not $n+1$ or more linearly independent vectors in it.
Secondly, we show 2) $\Longrightarrow$ 3). Since im $A_{n-1} =$ im A_n, im $A_{n-1} =$ im A_n $= \cdots =$ im A holds. Therefore, the dimension of im A_r is n for $r \leq n-1$. On the other hand, by Corollary (6-A.35) and Lemma (6-B.9), ker $P_s = 0$ for $s \leq n-1$. Consequently, the dimension of im $P_s \cdot A \cdot J_r$ is n. Therefore, the rank of partial Input/Output Matrix $(I/O)_{a\ (r,s)}$ corresponding to im $P_s \cdot A \cdot J_r$ is n.
Lastly, we show 3) $\Longrightarrow$ 1). Since the rank of the Input/Output Matrix $(I/O)_a$ is n, the range im A of the linear input/output map A corresponding to $(I/O)_a$ is n dimensional. By im $A = \{S_l^i(\chi(u)); u \in U, i \in N\}$ and Corollary (6-A.35), 1) is obtained.

6.7.B.5 Realization Procedure for Pseudo Linear Systems

This section is prepared for the proof of theorem for realization procedure (6.20).

(6-B.19) Proof of Theorem (6.20)
Let $R(\chi) = \{S_l^i(\chi(u)); u \in U, i \in N\}$. By Theorem (6.6), $((\ll S_l^N(\chi(U)) \gg, S_l), \chi, 0, a(1))$ is a canonical Pseudo Linear System that realizes a time-invariant input response map $a \in F(\Omega, Y)$. The linearly independent vectors $\{S_l^i(\chi(u)); u \in U, i \in N, 1 \le i \le n\}$ satisfy $\ll \{S_l^i(\chi(u_{J_i})); u \in U, i \in N, 1 \le i \le n; (I_1, u_{J_1}) < (I_2, u_{J_2}) < \cdots < (I_n, u_{J_n})\} \gg = \ll R(\chi) \gg$. Let a linear map $T : \ll R(\chi) \gg \to K^n$ be $T \cdot S_l^i(\chi(u_{J_i})) = \mathbf{e_i}$ for any $i (1 \le i \le n)$. Then, by step 2), $T\chi = g_s$ holds and by step 3), $h_s \cdot T = 0$ holds. And by step 4), $F_s \cdot T = T \cdot F_s$ holds. Consequently, T is bijective and a Pseudo Linear System morphism : $((\ll S_l^N(\chi(U)) \gg, S_l), \chi, 0, a(1)) \to \sigma_s = ((K^n, F_s), g_s, h_s, a(1))$.
By Corollary (6.9), the behavior of σ_s is a. It follows from the choice of $\{S_l^i(\chi(u_{J_i})); u \in U, i \in N, (I_1, u_{J_1}) < (I_2, u_{J_2}) < \cdots < (I_n, u_{J_n})$ for $i (1 \le i \le n\}$ and the determination of map T implies that σ_s is the quasi-reachable standard system.

6.7.C Partial Realization

In this Appendix, we give proofs for theorems and propositions stated in section 6.4. See Appendicies 6.7.A and 6.7.B for details of notions and notations.

6.7.C.1 Free Motions with an Affine Map

Set $A(N \times U, K)_p := \{\sum_{(q,u)} \lambda(q,u)\mathbf{e_{(q,u)}} \in A(N \times U, K), q \le p$ for some $p \in N\}$. J_p is the canonical injection : $A(N \times U, K)_p \to A(N \times U, K)$.
Let $H_q = P_q \cdot H$, where P_q is the canonical surjection : $F(N, Y) \to F(N_q, Y)$.

(6-C.1) Definition
If a free motion with an affine map $((X, F), g)$ satisfies
$X = \ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j} g(\omega(j)); \omega \in \Omega_p)\} \gg$, then $((X, F), g)$ is said to be p-quasi reachable.

Remark: Note that $((X, F), g)$ is p-quasi reachable if and only if $G_p := GJ_p : A(N \times U, K)_p \to X$ is surjective. Where G is the linear input map $: A(N \times U, K) \to (X, F)$ corresponding to $((X, F), g)$.

(6-C.2) Proposition

If a linear sub space S of $A(N \times U, K)_{p+1}$ satisfies the next two conditions, then there uniquely exists an ideal $\underline{S} \subseteq A(N \times U, K)$ such that $\underline{S} \cap A(N \times U, K)_{p+1} = S$ and $A(N \times U, K)_{p+1}/S$ is isomorphic to $A(N \times U, K)/\underline{S}$. Moreover, a free motion with an affine map $(A(N \times U, K)/\underline{S}, \bar{S_r}), [\eta])$ is p-quasi-reachable.

Where $\bar{S_r}$ is given by $\bar{S_r}(\lambda + \underline{S}) = S_r\lambda + \underline{S}$ for $\lambda \in A(N \times U, K)$ and $[\eta]$ is a map $: U \to A(N \times U, K)/\underline{S}; u \mapsto e_{(0,u)} + \underline{S}$.

condition 1 : $\lambda \in A(N \times U, K)_p \cap S$ implies $S_r\lambda \in S$.

condition 2 : There exist coefficients $\lambda(q, u') \in K$ such that $\mathbf{e}_{(\mathbf{p+1},\mathbf{u})} - \sum_{(q,u')} \lambda(q, u')\mathbf{e}_{(\mathbf{q},\mathbf{u'})} \in S$ for $q \le p$ and any $u \in U$.

[proof] Let $J_{(p,p+1)} : A(N \times U, K)_p \to A(N \times U, K)_{p+1}$ be the canonical injection and $\pi_S : A(N \times U, K)_{p+1} \to A(N \times U, K)_{p+1}/S$ be the canonical surjection. Then the condition 2 implies that a composition map $\pi_S \cdot J_{(p,p+1)}$ is surjective, and the condition 1 implies that $S_r\lambda \in S$ holds for any $\lambda \in S$. Therefore, by setting $\bar{S_r}(\lambda + S) = S_r\lambda + S$ for any $\lambda \in A(N \times U, K)_{p+1}$, we can uniquely define a map $\bar{S_r} \in L(A(N \times U, K)_{p+1}/S)$. $((A(N \times U, K)_{p+1}/S, \bar{S_r}), [\eta])$ is a free motion with an affine map and p-quasi-reachable.

Where $[\eta]$ is a map $: U \to A(N \times U, K)_{p+1}/S; u \mapsto e_{(0,u)} + S$. Then a linear input map $G : (A(N \times U, K), S_r) \to (A(N \times U, K)_{p+1}/S, \bar{S_r})$ corresponding to $((A(N \times U, K)_{p+1}/S, \bar{S_r}), [\eta])$ is uniquely determined by Proposition (6-A.23). Setting $G_{p+1} := GJ_{p+1}$, ker $G_{p+1} = S$ holds and $\underline{S} :=$ ker G satisfies $\underline{S} \cap A(N \times U, K)_{p+1} = S$. Since G is a linear input map, $\underline{S}$ is an invariant sub space under S_r. Moreover, the surjection of G implies that $((A(N \times U, K)_{p+1}/S, \bar{S_r}), [\eta])$ is isomorphic to $((A(N \times U, K)/\underline{S}, \tilde{S_r}), [\eta])$. Hence, $((A(N \times U, K)/\underline{S}, \tilde{S_r}), [\eta])$ is p-quasi-reachable. The uniqueness of $\underline{S}$ is obtained by the one of $\bar{S_r}$ and G.

6.7.C.2 Free Motions with a Readout Map

Set $F(N_q, Y) := \{\gamma \in F(N, Y); \gamma : N_q \to Y\}$, let P_q be the canonical surjection $: F(N, Y) \to F(N_q, Y); \gamma \mapsto [; t \mapsto \gamma(t)]$, and define $\underline{S_l}$ by setting $\underline{S_l} : F(N_q, Y) \to F(N_{q-1}, Y); \gamma \mapsto \underline{S_l}\gamma[; t \mapsto \gamma(t+1)]$.

(6-C.3) Definition

If a free motion with a readout map $((X, F), h)$ satisfies that $hF^t x = 0$ for any $t \in N_q$ implies $x = 0$, it is said to be q-observable.

Remark: Note that $((X, F), h)$ is q-observable if and only if a linear map $H_q := P_q \cdot H$ is injective. Where H is a linear observation map corresponding to $((X, F), h)$ and P_q is the canonical surjection : $F(N, Y) \to F(N_q, Y)$.

(6-C.4) Proposition

If a sub space Z of $F(N_{q+1}, Y)$ satisfies the next two conditions, then there uniquely exists a free motion (X, S_l) such that a map $P_{q\ X} : X \to Z$ is isomorphic. Where $P_{q\ X}$ is a restriction of the canonical surjection $P_q : F(N, Y) \to F(N_q, Y)$ to X, and a free motion with a readout map $((X, S_l), 0)$ is q-observable.

condition 3 : A composition map $\pi \cdot j : Z \xrightarrow{j} F(N_{q+1}, Y) \xrightarrow{\pi} F(N_q, Y)$ is injective.

condition 4 : im $(\underline{S_l} \cdot j) \subseteq$ im $(j \cdot \pi)$ holds in the sense of $F(N_q, Y)$.

Where π is the canonical surjection.

[proof] By conditions 3 and 4, we can define $Fz = (\pi \cdot j)^{-1} \underline{S_l} \cdot jz$ for any $z \in Z$. Then $F \in L(Z)$. Therefore, $((Z, F), 0)$ is an observable free motion with a readout map. Where 0 is a map : $Z \to Y; \gamma \mapsto \gamma(0)$. Injection of $\pi \cdot j$ implies that $((Z, F), 0)$ is q-observable. It follows that the linear observation map H corresponding to $((Z, F), 0)$ is injective. Set $X :=$ im H, a map $H^{-1} : X \to Z$ is clearly the restriction of the map $P_q : F(N, Y) \to F(N_q, Y)$ to X. An equation $0 = 0H$ implies that $((X, S_l), 0)$ is isomorphic to $((Z, F), 0)$ in the sense of free motion with a readout map. Therefore, $((X, S_l), 0)$ is q-observable. A uniqueness of X is obtained by the uniqueness of F and H.

6.7.C.3 Partial Realization Problem

We can consider a partial linear input/output map $A_{(p, \underline{N} - p)} : A(N \times U, K)_p \to F(N_{\underline{N} - p}, Y)$ for $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ the same as the linear input/output map $A : (A(N \times U, K), S_r) \to (F(N, Y), S_l)$ considered for $a \in F(\Omega, Y)$ in Appendix 6.7.A.

(6-C.5) Lemma

Let $A_{(p,\underline{N}-p)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then the following diagrams commute.

1)

$$\begin{array}{ccc} A(N \times U, K)_p & \xrightarrow{A_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p}, Y) \\ \underline{i} \downarrow & & \downarrow \pi \\ A(N \times U, K)_{p+1} & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1}, Y) \end{array}$$

Where $\underline{i}$ is a canonical injection and π is a canonical surjection.

2)

$$\begin{array}{ccc} A(N \times U, K)_p & \xrightarrow{A_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p}, Y) \\ S_r(u) \downarrow & & \downarrow \underline{S_l} \\ A(N \times U, K)_{p+1} & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1}, Y) \end{array}$$

[proof] These can be obtained by direct calculation.

(6-C.6) Proposition

Let $A_{(p_1,\underline{N}-p_1)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and p_2 be any integers such that $0 \leq p_2 \leq p_1 < \underline{N}$.
If im $A_{(p_2+1,\underline{N}-p_2-1)} =$ im $A_{(p_2,\underline{N}-p_2-1)}$, then im $A_{(p_1,\underline{N}-p_1)} =$ im $A_{(p_2,\underline{N}-p_1)}$ holds.

[proof] Note that this proposition holds if and only if im $A_{(p_2+1,\underline{N}-p_2-1)} =$ im $A_{(p_2,\underline{N}-p_2-1)}$ implies im $A_{(p_2+1,\underline{N}-p_2-1-n)} =$ im $A_{(p_2,\underline{N}-p_2-1-n)}$ holds for any non negative integer n. Therefore, we prove the latter by the inductive method. When $n = 0$, it holds by assumption.
Let's assume it holds for $n = k$, i.e. assume that im $A_{(p_2+1+k,\underline{N}-p_2-1-k)} =$ im $A_{(p_2,\underline{N}-p_2-1-k)}$. By assumption, there exist $m_j \leq p_2$ and $\lambda(m_j, u_j) \in$

$K(1 \le j \le m)$ such that $\underline{a}(u^{p_2+1+k+t+1}) - a(u^{p_2+1+k+t}) = \sum_{j=1}^{m} \lambda(m_j, u_j) \times (\underline{a}(u_j^{m_j+t+1}) - \underline{a}(u_j^{m_j+t})$ in sense of $F(N_{\underline{N}-p_2-1-k}, Y)$. Hence $\underline{S_l}(\underline{a}(u^{p_2+k+t+1}) - u^{p_2+k+t})) = \sum_{j=1}^{m} \lambda(m_j, u_j)\underline{S_l}(\underline{a}(u_j^{m_j+t}) - \underline{a}(u_j^{m_j+t-1}))$ hold. Therefore, im $A_{(p_2+1+k+1,\underline{N}-p_2-1-k-1)} =$ im $A_{(p_2+1,\underline{N}-p_2-1-k-1)}$ holds. On the other hand, im $A_{(p_2+1,\underline{N}-p_2-1)} =$ im $A_{(p_2,\underline{N}-p_2-1)}$ is equivalent to im $A(p_2+1, j) =$ im $A(p_2, j)$ for any $j \le \underline{N} - p_2 - 1$. Therefore, im $A_{(p_2+1+k+1,\underline{N}-p_2-1-k-1)} =$ im $A_{(p_2,\underline{N}-p_2-1-k-1)}$ holds. The equation holds for $n = k+1$.

(6-C.7) Proposition

Let $A_{(,)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. For p_1 and p_2 be any integers such that $0 \le p_2 < p_1 < \underline{N}$.
If ker $A_{(p_1,\underline{N}-p_1)} =$ ker $A_{(p_1,\underline{N}-p_1-1)}$ holds, then ker $A_{(p_2,\underline{N}-p_1-1)}$ $=$ ker $A_{(p_2,\underline{N}-p_2)}$ holds.

[proof] Note that this proposition holds if and only if ker $A_{(p_1,\underline{N}-p_1)} =$ ker $A_{(p_1,\underline{N}-p_1-1)}$ implies ker $A_{(p_1-n,\underline{N}-p_1-1)} =$ ker $A_{(p_1-n,\underline{N}-p_1+n)}$ for any n in $0 \le n \le p_1$. Therefore, we prove the latter by the inductive method. When $n = 0$, it holds by assumption.
Let's assume it holds for $n = k$, i.e., assume that ker $A_{(p_1-k,\underline{N}-p_1-1)} =$ ker $A_{(p_1-k,\underline{N}-p_1+k)}$. By assumption, there exist $t_j \le \underline{N}-p_1-1$ and $\lambda(s_j, u_j) \in K$ such that $\underline{a}(u^{\underline{N}-p_1+k+2}) - \underline{a}(u^{\underline{N}-p_1+k+1}) = \sum_{j=1}^{m} \lambda(s_j, u_j)$ $\times (\underline{a}(u_j^{t_j+1}) - \underline{a}(u_j^{t_j}))$.
Then, ker $A_{(p_1-k-1,\underline{N}-p_1)} = A_{(p_1-k-1,\underline{N}-p_1+k+1)})$ holds. On the other hand, if we note that ker $A_{(p_1,\underline{N}-p_1)} =$ ker $A_{(p_1,\underline{N}-p_1-1)}$ is equivalent to ker $A_{(i,\underline{N}-p_1)}$ $=$ ker $A_{(i,\underline{N}-p_1-1)}$ for any i in $0 \le i \le p_1$, ker $A_{(p_1-k-1,\underline{N}-p_1-1)}$ $=$ ker $A_{(p_1-k-1,\underline{N}-p_1+k+1)}$ holds. Therefore, the condition's equation holds for $n = k+1$.

(6-C.8) Lemma

For a partial linear input/output map $A_{(,)}$ corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and a Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$, the next matters hold. Where $G_p := G \cdot J_p$, $H_q := P_q \cdot H$ for the linear input map G corresponding to g and the linear output map H corresponding to h. $A_{(p,q)} := H_q \cdot J_p$.
1) σ is a partial realization of $\underline{a}$ if and only if the following figure commutes for any p such that $0 \le p < \underline{N}$.
2) σ is a natural partial realization of $\underline{a}$ if and only if the following figure commutes, G_p is surjective and $H_{\underline{N}-p-1}$ is injective for some p such that $0 \le p < \underline{N}$.

$$
\begin{array}{ccccc}
A(N\times U,K)_p & \xrightarrow{G_p} & X & \xrightarrow{H_{\underline{N}-p}} & F(N_{\underline{N}-p},Y) \\
\Big\downarrow S_r & & \Big\downarrow F & & \Big\downarrow \underline{S_l} \\
A(N\times U,K)_{p+1} & \xrightarrow{G_{p+1}} & X & \xrightarrow{H_{\underline{N}-p-1}} & F(N_{\underline{N}-p-1},Y)
\end{array}
$$

[proof] These can be obtained by definition of the partial and natural partial realization.

(6-C.9) Proof of Theorem (6.23)
We prove the theorem by rewriting the conditions of partial Input/Output Matrix in Theorem (6.23) to partial linear input/output map $A_{(,)}$ corresponding to $\underline{a} \in F(N_{\underline{N}}, Y)$. By using Propositions (6-C.6) and (6-C.7), the conditions of Input/Output Matrix can be equivalently changed to the following equations (1) & (2):
(1) im $A_{(p,\underline{N}-p-1)} =$ im $A_{(p+1,\underline{N}-p-1)}$.
(2) ker $A_{(p,\underline{N}-p)} =$ ker $A_{(p,\underline{N}-p-1)}$.
Therefore, we will prove the theorem by using (1) and (2).
First, we show that the above equations (1) & (2) are necessary. Let $\sigma = ((X, F), g, h, h^0)$ be a natural partial realization of $\underline{a} \in F(N_{\underline{N}}, Y)$, then σ is p-quasi-reachable and q-observable for some p and q such that $p + q < \underline{N}$. Let G be the linear input map corresponding to g and H be the linear observation map corresponding to h, and let $p \le p'$ and $q \le q'$, then $G_{p'} := G \cdot J_{p'}$ is onto, $H_{q'} := P_{q'} \cdot H$ is one-to-one. Therefore, $A_{(p',q')} := H_{q'} \cdot J_{p'}$ satisfies equations (1) and (2).
Next, we show that the equations (1) & (2) are sufficient.
Set $S :=$ ker $A_{(p+1,\underline{N}-p-1)}$ and $Z :=$ im $A_{(p,\underline{N}-p)}$. Then equation (2) implies that a composition map $\pi \cdot j : Z \overset{j}{\to} F(N_{\underline{N}-p}, Y) \overset{\pi}{\to} F(N_{\underline{N}-p-1}, Y)$ is injective. Where π and j are the same as in Proposition (6-C.4). Therefore, Z satisfies condition 3 in Proposition (6-C.4). Equation (1) implies that there exists $\mathbf{e}_{(\mathbf{l},\mathbf{u_i})} \in A(N \times U, K)_p$ such that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)}) = A_{(p+1,\underline{N}-p-1)}(\sum_i \lambda(l, u_i) e_{(l,u_i)})$ for any $u \in U$.

By Lemma (6-C.5), we obtain that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)} - \sum_i \lambda(l,u_i)e_{(l,u_i)}) = 0$, and $e_{(p+1,u)} - \sum_i \lambda(l,u_i)e_{(l,u_i)} \in S$ holds.
This implies that S satisfies the condition 2 in Proposition (6-C.2). Let $\underline{j}$ be the canonical injection : $A_{(p,\underline{N}-p-1)} \to F(N_{\underline{N}-p-1}, Y)$ and π is the same as in Proposition (6-C.3), $B := (\underline{j})^{-1} \cdot \pi \cdot j : Z \to \text{im } A_{(p,\underline{N}-p-1)}$ is a bijective linear map by (2) in Proposition (6-C.2). When we consider the bijective linear map $A^b := A^b_{(p+1,\underline{N}-p-1)} : A(N \times U, K)_{p+1}/S \to \text{im } A_{(p+1,\underline{N}-p-1)}$ associated with $A_{(p+1,\underline{N}-p-1)} : A(N \times U, K)_{p+1} \to F(N_{\underline{N}-p-1}, Y)$, equation (2) implies that a linear map $B^{-1} \cdot A^b$ is a bijective linear map : $A(N \times U, K)_{p+1}/S \to Z$. For any $\lambda \in A(N \times U, K)_p \cap S$, $A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by injection of $B^{-1} \cdot A^b$. Therefore, $A_{(p+1,\underline{N}-p-1)}(S_r\lambda) = \underline{S_l} A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by using 2) in Lemma (6-C.5). This implies that $S_r\lambda \in S$. Therefore, S satisfies the condition 1 in Proposition (6-C.2). Then Proposition (6-C.2) implies that a free motion with an affine map $(A(N \times U, K)_{p+1}/S, \bar{S}_r), [\eta]))$ is p-quasi-reachable. Where $[\eta]$ is a map : $U \to A(N \times U, K)_{p+1}/S; u \mapsto e(0,u) + S$. Here, equation (1) implies that there exists $x \in \text{im } A_{(p,\underline{N}-p-1)}$ such that $\underline{j}(x) = \underline{S_l} \cdot j(z)$ for any $z \in Z$. Moreover, by surjection of B, there exists $z' \in Z$ such that $B(z') = x$. Hence, $\underline{S_l} \cdot j(z) = \underline{j}(x) = \underline{j} \cdot B(z') = \pi \cdot j(z')$, which implies that $\text{im } (\underline{S_l} \cdot j) \subseteq \text{im } (\pi \cdot j)$. It follows that Z satisfies condition 4 in Proposition (6-C.4) and $((Z,F),0)$ is $(\underline{N}-p-1)$-observable. We can also show that $B^{-1} \cdot A^b$ is a free motion morphism : $A(N \times U, K)_{p+1}/S, \bar{S}_r) \to (Z, F)$, and that a Pseudo Linear System $\sigma_1 = (A(N \times U, K)_{p+1}/S, \bar{S}_r), [\eta], 1 \cdot B^{-1} \cdot A^b, a(1))$ is isomorphic to a Pseudo Linear System $\sigma_2 = ((Z,F), B^{-1} \cdot A^b \cdot [\eta], 0, a(1))$. It follows that σ_1 and σ_2 are the natural partial realizations of $\underline{a} \in F(N_{\underline{N}}, Y)$. Hence, the natural partial realization of $\underline{a} \in F(N_{\underline{N}}, Y)$ exist.

(6-C.10) Lemma
Two canonical Pseudo Linear Systems are isomorphic if and only if their behaviors are the same.

[proof] This can be obtained from Theorem (6.8) and Corollary (6.9).

(6-C.11) Proof of Theorem (6.24)
Let $A_{(\ ,\ )}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. To prove necessity, we assume existence of the natural partial realization of $\underline{a}$. Let Theorem (6.23) hold for integers p and p' that are different. Namely, (1): $\text{im } A_{(p,\underline{N}-p-1)} = \text{im } A_{(p+1,\underline{N}-p-1)}$ (2): $\ker A_{(p,\underline{N}-p)} = \ker A_{(p,\underline{N}-p-1)}$

(3): im $A_{(p',\underline{N}-p'-1)} = \text{im } A_{(p'+1,\underline{N}-p'-1)}$ (4): ker $A_{(p',\underline{N}-p')} = \ker A_{(p',\underline{N}-p'-1)}$ Then Propositions (6-C.6) and (6-C.7) imply that the dimension of $Z = \text{im } A_{(p,\underline{N}-p-1)}$ is equal to one of $Z' = \text{im } A_{(p',\underline{N}-p'-1)}$. Let σ and σ' be the natural partial realizations of $\underline{a}$ whose state space are Z and Z' respectively and which can be obtained by the same procedure in (6-C.9). Then σ is clearly isomorphic to σ' and the behavior of σ is equal to one of σ' by Lemma (6-C.10). This implies that the behavior of the natural partial realization is always the same regardless of different integers p and p'. Therefore, the natural partial realization of $\underline{a}$ is unique modulo isomorphism by Lemma (6-C.10).
Next, we show sufficiency by the contrapositive. We assume that there does not exist a natural partial realization of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then minimum dimensional partial realization σ of $\underline{a}$ is p-quasi-reachable and q-observable for $p+q \geq \underline{N}$. It cannot be quasi-reachable within $p-1$ and not be observable within $q-1$. Then, there exists a state x in σ such that x can be at first reachable by an input ω with length p. The remaining data of $F(\Omega_{\underline{N}-p-1)}, Y)$ can't determine a new state Fx, because of $\underline{N} - p - 1 < q$. Therefore, we can't determine the transition matrix F uniquely by q-observability. This implies that the minimum dimensional realization of $\underline{a}$ is not unique.

(6-C.12) Proof of Theorem (6.25)
Let's consider the natural partial realization $\sigma_2 = ((Z, F), B^{-1} \cdot A^b \cdot [\eta], \underline{0}, a(\mathit{1}))$ of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ given in (6-C.9). Then we can obtain the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s, h_s, a(\mathit{1}))$ from σ_2 in the same manner as theorem for a realization procedure (6.20).

6.7.D Real-Time Partial Realization Problem

(6-D.1) Proof of Lemma (6.28)
Note that jth component of row vectors $\underline{S_l}^i(\chi(u_1)) \in K^{L-1}$ is $(\underline{S_l}^i(\chi(u_1))(j-1) = \underline{a}(u_1^{i+j}) - \underline{a}(u_1^{i+j-1})$. Then $\{\underline{a}(u_1^k|\omega_1)$; any $k \in N\}$ can be uniquely determined by linear dependence. Since $\underline{a}$ is time-invariant, $\underline{a}(u_1^{k-1}|u_2|\omega_1) - \underline{a}(u_1^k|\omega_1) + \underline{a}(u_1^k) - \underline{a}(u_1^{k-1}) = \underline{a}(u_1^{k-1}|u_2) - \underline{a}(u_1^{k-1}) = (\underline{S_l}^i(\chi(u_2)))(j-1)$ can be obtained for $i + j = k$. Therefore, if we add a further input $\omega_2 = u_1^{L_2+L-1}|u_2$, step 2) can be inspected. Since jth component of row vectors $\underline{S_l}^i(\chi(u_2)) \in K^{L-1}$ is $(\underline{S_l}^i(\chi(u_2))(j-1) = a(u_1^{i+j}|u_2|\omega_1) - a(u_1^{i+j}|\omega_1) + a(u_1^{i+j}) - a(u_1^{i+j-1})$, $\{\underline{a}(u_1^k|\omega_2|\omega_1)$; any $k \in N\}$ can be uniquely determined by linear dependence. Since $\underline{a}$ is time-invariant, $\underline{a}(u_1^{k-1}|u_3|\omega_2|\omega_1) - \underline{a}(u_1^k|\omega_2|\omega_1) + \underline{a}(u_1^k) - \underline{a}(u_1^{k-1}) = \underline{a}(u_1^{k-1}|u_3) - \underline{a}(u_1^{k-1}) = (\underline{S_l}^i(\chi(u_3)))(j-1)$ can be obtained

for $i+j=k$. Therefore, if we add a further input $\omega_3 = u_1^{L_3+L-1}|u_3$, step 3) can be inspected. Since j-th component of row vectors $\underline{S_l}^i(\chi(u_3)) \in K^{L-1}$ is $(\underline{S_l}^i(\chi(u_3))(j-1) = \underline{a}(u_1^{k-1}|u_3) - \underline{a}(u_1^{k-1}) = \underline{a}(u_1^{k-1}|u_3|\omega_2|\omega_1) - \underline{a}(u_1^k|\omega_2|\omega_1) + \underline{a}(u_1^k) - \underline{a}(u_1^{k-1})$ for $i+j=k$, $\{\underline{a}(u_1^k|\omega_3|\omega_2|\omega_1)$; any $k \in N\}$ can be uniquely determined by linear dependence. The same manner as the above can be done, then a partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ is obtained. Since the physical object is less than L dimensional, the partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ contain the whole input/output data by (6-C.9).

7 Almost Linear Systems

Almost Linear Systems are presented with the following main theorem. The main theorem says that for any input/output map with causality, time-invariance and affinity, there exist at least two canonical Almost Linear Systems which realize it and any two canonical Almost Linear Systems with the same behavior are isomorphic.

Secondly, details of finite dimensional Almost Linear Systems are investigated. A criterion for being canonical of finite dimensional Almost Linear Systems is given. Representation theorems of isomorphic classes of canonical Almost Linear Systems are given. We give a criterion for the behavior of finite dimensional Almost Linear Systems. We derive a procedure to obtain a canonical Almost Linear System.

Thirdly, these partial realization problem is discussed according to the above results. Existence of minimum partial realization is easily presented. It hardly ever happens for minimum partial realizations to be unique up to isomorphism. To solve the uniqueness problem, we introduce the notion of natural partial realizations.

The main results for partial realization are the followings:
1) A necessary and sufficient condition for the existence of the natural partial realizations is given by the rank condition of finite sized Input/Output Matrix.
2) The existence condition of natural partial realization is equivalent to the uniqueness condition of minimum partial realizations.
3) An algorithm to obtain a natural partial realization from a partial time-invariant, affine input response map is given.
Moreover, for a time-invariant, affine input response map, we can discuss the real time partial realization problem. Namely, by a single experiment, we find a mathematical model from on-line data. An algorithm to obtain a partial realization from the data is given, if we know that the physical object is finite dimensional before hand.

7.1 Time-Invariant, Affine Input Response Maps (Input Response Maps with Time-Invariance and Affinity)

In this chapter we consider input/output maps $a \in F(\Omega, Y)$ which satisfy the following time-invariant condition and affinity condition. They are said to be time-invariant, affine input response maps. Where U is a linear space

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throughout this chapter. We could discuss the case where multi-inputs are fed, i.e., $U = K^m$, but conveniently, we discuss a case where one-input is fed, i.e., $U = K$. And Y is a linear space over the field K.

(7.1) Definition

If an input response map a satisfies the following time-invariant and affinity condition, then a is said to be a time-invariant, affine input response map.
Time-invariant condition:
$a(\omega_1|\omega) - a(\omega_1) = a(\bar{\omega}_1|\omega) - a(\bar{\omega}_1)$
for any ω, and $\omega_1, \bar{\omega}_1$ such that $|\omega_1| = |\bar{\omega}_1|$.
Affinity condition:
$a : \Omega \to Y$ is an affine map, i.e.,
$a(\omega + \bar{\omega}) + a(0^{|\omega|}) = a(\omega) + a(\bar{\omega})$
$a(\lambda\omega) = \lambda a(\omega) + (1 - \lambda)a(0^{|\omega|})$
for any $\omega, \bar{\omega} \in \Omega, |\omega| = |\bar{\omega}|$ and $\lambda \in K$.

(7.2) Definition

For any time-invariant, affine input response map $a \in F(\Omega, Y)$, a function $GI_a : \{0, 1\} \to F(N, Y); u \mapsto GIa(u)[; t \mapsto a(u^{t+1}) - a(u^t)]$ is said to be a modified impulse response of a.

(7.3) Representation Theorem

For any time-invariant, affine input response map $a \in F(\Omega, Y)$, there exists uniquely the modified impulse response of a by the following equation. This correspondence is bijective.

$a(\omega) = a(1) + \sum_{j=1}^{|\omega|}(\omega(j))(GI_a(1)(|\omega| - j + 1)) + (1 - \omega(j))(GI_a(0)(|\omega| - j + 1))$ for any $\omega \in \Omega$.

[proof] This theorem is obtained by direct calculation.

7.2 Almost Linear Systems

(7.4) Definition

A system given by the following equations is written as a collection $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$ and it is said to be an Almost Linear System.

$$\begin{cases} x(t+1) &= Fx(t) + g^0 + \bar{g}\omega(t+1) \\ x(0) &= 0 \\ \gamma(t) &= h^0 + hx(t) \end{cases}$$

Where X is a linear space over the field K, F is a linear operator on X and $\omega(t) \in U$ for any $t \in N$. And $g^0, g \in X$, h is a linear operator : $X \to Y$ and $h^0 \in Y$.
The input response map $a_\sigma :\to Y; \omega \mapsto h^0 + h(\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j))$ is said to be the behavior of σ.
For a time-invariant, affine input response map $a \in F(\Omega, Y)$, σ that satisfies $a_\sigma = a$ is called a realization of a.
Note that the behavior a_σ of an Almost Linear System σ is a time-invariant, affine input response map.
An Almost Linear System σ is said to be quasi-reachable if the linear hull of the reachable set $\{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j)); \omega \in \Omega\}$ is equal to X. An Almost Linear System σ is called observable if $hF^m x_1 = hF^m x_2$ for any $m \in N$ implies $x_1 = x_2$.
An Almost Linear System σ is called canonical if σ is quasi-reachable and observable.

Remark 1: The $x(t)$ in the system equation of σ is the state that produces output values of a_σ at the time t by adding h^0, namely the state $x(t)$ and linear operator $h : X \to Y$ generates the output value $a_\sigma(t) = h^0 + hx(t)$.
Remark 2: It is meant for σ to be a faithful model for the time-invariant, affine input response map $a \in F(\Omega, Y)$ such that σ realizes a.
Remark 3: Note that a canonical Almost Linear System $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$ is a system that has the most reduced state space X among systems that have the behavior a (see Corollary (7-A.8), Proposition (7-A.14), Corollary (7-A.15), Propositions (7-A.20) and (7-A.21), Proposition (7-A.24) and Theorem (7-A.25)).

(7.5) Example
$A(N\times\{0,1\}, K) := \{\lambda = \sum_{n,u} \lambda(n,u)\mathbf{e}_{(\mathbf{n},\mathbf{u})}$ (finite sum) ; $n \in N, u \in \{0,1\}\}$. Where $\mathbf{e}_{(\mathbf{n},\mathbf{u})}$ is given by the following equations for $n, n' \in N$ and $u, u' \in \{0,1\}$. If $n = n'$ and $u = u'$ imply $\mathbf{e}_{(\mathbf{n},\mathbf{u})}(n', u') = 1$. If $n \neq n'$ or $u \neq u'$ imply $\mathbf{e}_{(\mathbf{n},\mathbf{u})}(n', u') = 0$. Then $A(N \times \{0,1\}, K)$ is clearly a linear space. Let S_r be $S_r\mathbf{e}_{(\mathbf{n},\mathbf{u})} = \mathbf{e}_{(\mathbf{n+1},\mathbf{u})}$, then $S_r \in L(A(N \times \{0,1\}, K))$ and S_r is irrelevant to the input value's set $\{0,1\}$. S_r is a right shift operator. Let $\bar{\eta} := e_{(0,1)} - e_{(0,0)}$ and let a linear map $\bar{a} : A(N\times\{0,1\}, K) \to Y$ be $\bar{a}(\mathbf{e}_{(\mathbf{n},\mathbf{u})}) = a(u^{n+1}) - a(u^n)$

for any time-invariant, affine input response map $a \in F(\Omega, Y)$. Then a collection $((A(N \times \{0,1\}, K), S_r), \mathbf{e}_{(\mathbf{0},\mathbf{0})}, \bar{\eta}, \bar{a}, a(1))$ is a quasi-reachable Almost Linear System that realizes a.
Let $F(N, Y) := \{$ any function $f : N \to Y\}$. Let $S_l\gamma(t) = \gamma(t+1)$ for any $\gamma \in F(N, Y)$ and $t \in N$, then $S_l \in L(F(N, Y))$. Let a map $\chi^0 \in F(N, Y)$ be $(\chi^0)(t) := a(\omega|0) - a(\omega)$ and $\bar{\chi} \in F(N, Y)$ be $(\bar{\chi})(t) := a(\omega|1) - a(\omega|0)$ for any $t \in N$, a time-invariant, affine input response map $a \in F(\Omega, Y)$ and $\omega \in \Omega$ such that $|\omega| = t$. Moreover, let a linear map 0 be $F(N, Y) \to Y; \gamma \mapsto \gamma(0)$. Then a collection $((F(N, Y), S_l), \chi^0, \bar{\chi}, 0, a(1))$ is an observable Almost Linear System that realizes a.

(7.6) Theorem
The following two Almost Linear Systems are canonical realizations of any time-invariant, affine input response map $a \in F(\Omega, Y)$.
1) $(A(N \times \{0,1\}, K)/_{=a}, \hat{S}_r), [e_{(0,0)}], \hat{\bar{\eta}}, \hat{\bar{a}}, a(1))$.
Where $A(N \times \{0,1\}, K)/_{=a}$ is a quotient space obtained by equivalence relation $\sum_{(n,u)} \lambda_1(n,u)\mathbf{e}_{(\mathbf{n},\mathbf{u})} = \sum_{(n',u')} \lambda_2(n',u')\mathbf{e}_{(\mathbf{n}',\mathbf{u}')} \iff$
$\sum_{(n,u)} \lambda(n,u)(a(u^{n+1}) - a(u^n)) = \sum_{(n,u)} \lambda(n,u)(a(u^{n+1}) - a(u^n))$.
And $\hat{S}_r \in L(A(N \times \{0,1\}, K)/_{=a})$ is given by $\hat{S}_r[e(n,u)] = [e_{(n+1,u)}]$ for $[e_{(n,u)}] \in A(N \times \{0,1\}, K)/_{=a}$, and $\hat{\bar{\eta}} = [e_{(0,1)}] - [e_{(0,0)}]$, $\hat{\bar{a}}$ is given by $\hat{\bar{a}} : A(N \times \{0,1\}, K)/_{=a} \to Y; [e(n,u)] \mapsto a(u^{n+1}) - a(u^n)$.
2) $((\ll S_l^N(\chi(U)) \gg, S_l), \chi^0, \bar{\chi}, 0, a(1))$.
Where $\ll S_l^N(\chi(U)) \gg$ is the smallest linear space that contains $S_l^N(\chi(U))$ $:= \{S_l^i(\chi^0 + \bar{\chi}u); u \in K, i \in N,\ S_l^i(\chi^0 + \bar{\chi}u)(t) = (\chi(u)(t+1) = a(\omega|u) - a(\omega), \omega \in \Omega\}$.

[proof] See Corollary (7-A.15), Propositions (7-A.16), (7-A.21) and (7-A.24) and Corollary (7-A.26).

(7.7) Definition
Let $\sigma_1 = ((X_1, F_1), g_1^0, \bar{g}_1, h_1, h^0)$ and $\sigma_2 = ((X_2, F_2), g_2^0, \bar{g}_2, h_2, h^0)$ be Almost Linear Systems, then a linear operator $T : X_1 \to X_2$ is said to be an Almost Linear System morphism $T : \sigma_1 \to \sigma_2$ if T satisfies $TF_1 = F_2T$, $Tg_1^0 = g_2^0$, $T\bar{g}_1 = \bar{g}_2$ and $h_1 = h_2T$.
If $T : X_1 \to X_2$ is bijective, then $T : \sigma_1 \to \sigma_2$ is said to be an isomorphism.

(7.8) Realization Theorem of Almost Linear Systems
For any time-invariant, affine input response map $a \in F(\Omega, Y)$, there exist at least two canonical Almost Linear Systems that realize a. (Existence part)
Let σ_1 and σ_2 be any two canonical Almost Linear Systems which realize a time-invariant, affine input response map $a \in F(\Omega, Y)$, then there exists an isomorphism $T : \sigma_1 \to \sigma_2$. (Uniqueness part)

[proof] The first half is obvious from Theorem (7.6). The latter part is obtained by Corollary (7-A.15), Propositions (7-A.16), (7-A.21) and (7-A.24) and Remark in Lemma (7-A.30).

7.3 Finite Dimensional Almost Linear Systems

Based on the realization theory (7.8), we study structures of finite-dimensional Almost Linear Systems in this section. Main results can be stated in the following four steps:

First, we present conditions when finite dimensional Almost Linear System is canonical.

Secondly, we obtain the representation theorem for finite dimensional canonical Almost Linear Systems, namely, we show that there exist two standard systems as a representative in their equivalence classes. One is the quasi-reachable standard system, and the other is the observable standard system.

Thirdly, we give a criterion for the behavior of finite dimensional Almost Linear Systems. It is the rank condition of an infinite Input/Output Matrix.

Lastly, we give a procedure to obtain the quasi-reachable standard system that realizes a given time-invariant, affine input response map.
We will prove the above matters in Appendix 7.7.B.

(7.9) Corollary
Let T be an Almost Linear System morphism : $\sigma_1 \to \sigma_2$, then $a_{\sigma_1} = a_{\sigma_2}$ holds.

[proof] This is a direct calculation by the definition of the behavior and Almost Linear System morphism.

Following is a fact about finite dimensional linear spaces:

$<$ A n-dimensional linear space over the field K is isomorphic to K^n and $L(K^n, K^m)$ is isomorphic to $K^{m\times n}$ (see Halmos [1958]). $>$

Therefore, without loss of generality, we can consider n-dimensional Almost Linear System as $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$, where $F \in K^{n\times n}$, $g^0, \bar{g} \in K^n$ and $h \in K^{p\times n}$.

(7.10) So-called Linear Systems

Let a So-called Linear System be a linear system with a non-zero initial state.

$$\begin{cases} \underline{x}(t+1) & = & A\underline{x}(t) + \mathbf{b}\omega(t) \\ \underline{x}(0) & = & \underline{x}^0 \\ \lambda(t) & = & \mathbf{c}\underline{x}(t) \end{cases}$$

A So-called n-dimemsional Linear System with one-input and one-output described by the above equations can be changed to a So-called (n+1)-dimensional Linear System described by the following equations:

$$\begin{cases} x(t+1) & = & Fx(t) + \bar{g}\omega(t+1) \\ x(0) & = & x^0 \\ \gamma(t) & = & hx(t) \end{cases}$$

Where . $x(t) = \begin{bmatrix} \underline{x}(t) \\ \omega(t) \end{bmatrix}$, $F = \begin{bmatrix} A & \mathbf{b} \\ 0 & 0 \end{bmatrix}$, $\bar{g} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x^0 = \begin{bmatrix} \underline{x}^0 \\ \omega(0) \end{bmatrix}$, $h = \begin{bmatrix} c & d \end{bmatrix}$.

Note that the canonicality (controllable and observable) between two So-called Linear Systems given in (7.10) is preserved.

(7.11) From So-called Linear Systems to Almost Linear Systems

For any So-called Linear System given by the following equations, there exists an Almost Linear System $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$.

Where $g^0 = Fx^0 - x^0$, $h^0 = hx^0$.

$$\begin{cases} x(t+1) & = & Fx(t) + \bar{g}\omega(t+1) \\ x(0) & = & x^0 \\ \gamma(t) & = & hx(t) \end{cases}$$

Where $t \in N$, $x(t) \in X$.

(7.12) Proposition

Let $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ be the observable Almost Linear System. σ is given by the following system equations if and only if there exists $x^0 \in X$ such that $[F - I]x^0 = g^0$.

$$\begin{cases} x(t+1) &= Fx(t) + \bar{g}\omega(t+1) \\ x(0) &= 0 \\ \gamma(t) &= h^0 - hx^0 + hx(t) \end{cases}$$

(7.13) Theorem

An Almost Linear System $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ is canonical if and only if the following conditions 1) and 2) hold:

1) rank $[g^0, Fg^0, F^2g^0, \cdots, F^{n-1}g^0, \bar{g}, F\bar{g}, F^2\bar{g}, \cdots, F^{n-1}\bar{g}] = n$

2) rank $[h^T, (hF)^T, , , (hF^{n-1})^T] = n$.

[proof] See Proposition (7-B.6) and (7-B.13) in Appendix 7.7.B.

(7.14) Definition

Let a map $\| \, \|$: $N \times \{0,1\} \to N$ be : $(i,v) \mapsto \| (i,v) \| = m \times i + v$. Then $\| (i,v) \|$ is said to be a numerical value of $(i,v) \in N \times \{0,1\}$.

And we define totally ordered relation by this numerical value in $N \times \{0,1\}$. Namely, $(p, u_p) \leq (q, u_q) \iff \| (p, u_p) \| \leq \| (q, u_q) \|$.

(7.15) Definition

A canonical Almost Linear System $\sigma_s = ((K^n, F_s), , g_s^0, \bar{g}_s, h_s, h^0)$ is said to be a quasi-reachable standard system if a set $\{(I_i, J_i) \in N \times \{0,1\}, 1 \leq i \leq n\}$ given by $\mathbf{e_i} = F_s^{I_i}(g_s^0 + \bar{g}_s \cdot J_i)$ satisfies the following conditions:

1) $(I_1, J_1) < (I_2, J_2) < \cdots < (I_n, J_n)$ holds.

2) $I_i < i$

3) $F_s^p(g_s^0 + \bar{g}_s \cdot q) = \sum_{i=1}^{j} \alpha_i \mathbf{e_i}$ holds for any $(p,q) \in N \times \{0,1\}$ such that $(I_j, J_j) < (p,q) < (I_{j+1}, J_{j+1})$.

Where $\alpha_i \in K$ and $\mathbf{e_i} = [0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T$.

See Figure 7.1 for the quasi-reachable standard system.

(7.16) Representation Theorem for equivalence classes

For any finite dimensional canonical Almost Linear System, there exists a uniquely determined isomorphic quasi-reachable standard system.

[proof] See (7-B.16) in Appendix 7.7.B.

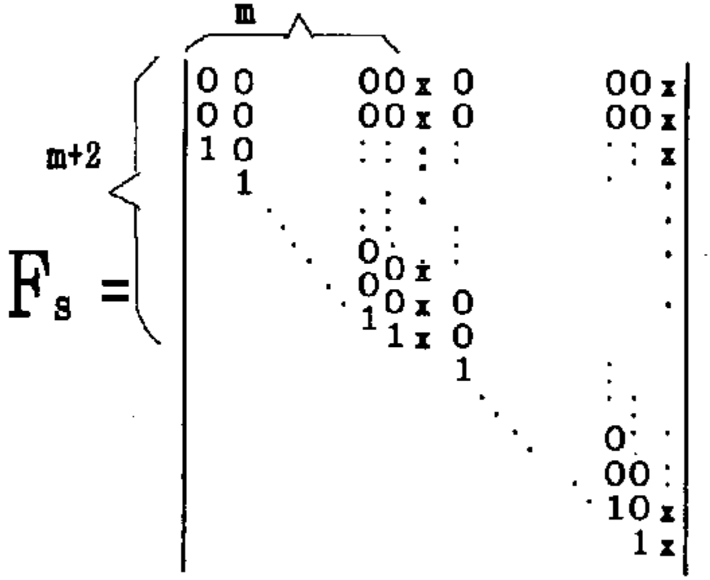

Fig. 7.1. F_s of the quasi-reachable standard system $\sigma_s = ((K^n, F_s), , g_s^0, \bar{g}_s, h_s, h^0)$

(7.17) Definition
A canonical Almost Linear System $\sigma_o = ((K^n, F_o), g_o^0, g_o, h_o, h^0)$ is said to be an observable standard system if $\mathbf{e_i}^T = h_o F_o^{i-1}$ holds for $1 \leq i \leq n$.
Where Y is K.

(7.18) Representation Theorem for equivalence classes
For any finite dimensional canonical Almost Linear System, there exists a uniquely determined isomorphic observable standard system.
Where Y is K.

[proof] See (7-B.17) in Appendix 7.7.B.

(7.19) Definition
For any time-invariant, affine input response map $a \in F(\Omega, Y)$, the corresponding linear input/output map $A : (A(N \times \{0,1\}, K), S_r) \to (F(N, Y), S_l)$ satisfies $A(\mathbf{e_{(s,u)}})(t) = a(u^{s+t+1}) - a(u^{s+t})$ for any $u \in \{0, 1\}$.
Therefore, A can be represented by the next infinite matrix $(I/O)_a$.
This $(I/O)_a$ is said to be an Input/Output Matrix of a.

$$(I/O)_{\substack{a\\t}} = \begin{pmatrix} & & (s,u) & \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \\ \cdots & \cdots & a(u^{s+t+1}) - a(u^{s+t}) & \\ & & & \end{pmatrix}$$

See Corollary (7-A.15) about the corresponding linear input/output map A.

(7.20) Theorem for existence criterion

For a time-invariant, affine input response map $a \in F(\Omega, Y)$, the following conditions are equivalent:

1) The time-invariant, affine input response map $a \in F(\Omega, Y)$ has the behavior of n-dimensional canonical Almost Linear System.

2) There exist n linearly independent vectors and no more than n linearly independent vectors in a set $\{S_l^i(\chi^0 + \bar{\chi}u); u \in \{0,1\}, i \in N, 1 \leq i \leq n\}$.

3) The rank of the Input/Output Matrix $(I/O)_a$ of a is n.

[proof] See (7-B.18) in Appendix 7.7.B.

(7.21) Theorem for a Realization Procedure

Let a time-invariant, affine input response map $a \in F(\Omega, Y)$ satisfy the condition of Theorem(7.20), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, \bar{g}_s, h_s, h^0)$ which realizes a can be obtained by the following procedure:

1) Select the linearly independent vectors $\{S_l^{I_i}(\chi^0 + \bar{\chi} \cdot J_i)\}$ of the set $\{S_l^i(\chi^0 + \bar{\chi} \cdot J_i); J_i \in \{0,1\}, I_i \in N, 1 \leq i \leq n\}$ in order of the numerical value $N \times \{0,1\}$. Let n:= rank $(I/O)_a$.

2) Let the state space be K^n. Let the map $g_s^0 = \mathbf{e_1}$ and $g_s := \mathbf{e_2} - \mathbf{e_1}$. Where $J_2 = 0$. If $J_2 \neq 0$, $g_s := \alpha \cdot \mathbf{e_1}$, $\alpha \in K$.

3) Let the output map $h_s = [a(J_1) - a(\mathit{1}), a(J_2^{I_2+1}) - a(J_2^{I_2}), \cdots, a(J_n^{I_n+1}) - a(J_n^{I_n})]$

4) Let f_i in $F_s := [f_1, f_2, \cdots, f_n] \in K^{n \times n}$ be $f_i := [f_{i,1} f_{i,2}, \cdots, f_{i,n}]$.
Where $S_l^{I_i+1}(\chi 0 + \bar{\chi} J_i) = \sum_{j=1}^n f_{i,j} S_l^{I_j}(\chi 0 + \bar{\chi} J_j)$, $f_{i,j} \in K$.

[proof] See (7-B.19) in Appendix 7.7.B.

7.4 Partial Realization Theory of Almost Linear Systems

Here we consider a partial realization problem by multi-experiment. Let $\underline{a}$ be an $\underline{N}$ sized time-invariant, affine input response map ($\underline{a} \in F(\Omega_{\underline{N}}, Y)$), where $\underline{N} \in N$ and $\Omega_{\underline{N}} := \{\omega \in \Omega; |\omega| \leq \underline{N}\}$. The $\underline{a}$ is said to be a partial time-invariant, affine input response map.

A finite dimensional Almost Linear System $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ is said to be a partial realization of $\underline{a}$ if $h^0 + h(\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j))) = \underline{a}(\omega)$ holds for any $\omega \in \Omega_{\underline{N}}$.

A partial realization problem of Almost Linear Systems can be stated as follows:
< For any given partial time-invariant, affine input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, find a partial realization σ of $\underline{a}$ such that the dimensions of state space X of σ is minimum, where the σ is said to be a minimal partial realization of $\underline{a}$. Moreover, show when the minimal realizations are isomorphic.>

In section 7.1, we have obtained the representation theorem for the time-invariant, affine input response maps. The theorem says that any time-invariant, affine input response map can be characterized by the modified impulse response.

Note that the modified impulse response $GI : \{0, 1\} \to F(N, Y)$ can be represented by $(GI(u)(t)) = a(u^{t+1}) - a(u^t)$ for $u \in \{0, 1\}$, $t \in N$ and the time-invariant, affine input response map $a \in F(\Omega, Y)$.

For any given partial time-invariant, affine input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, this correspondence can determine a partial modified impulse response GI : $\{0, 1\} \to F(N_{\underline{N}}, Y)$. Where $N_{\underline{N}} := \{1, 2, \cdots, \underline{N};$ for some $\underline{N} \in N\}$.

(7.22) Proposition
For any given time-invariant, affine input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, there always exists a minimal partial realization of it.

[proof] For any $\omega \notin \Omega_{\underline{N}}$, set $\underline{a}(\omega) = 0$. Then $\underline{a} \in F(\Omega, Y)$, and Theorem (7.20) implies that there exists a finite dimensional partial realization of $\underline{a}$. Therefore, there exists a minimal partial realization of it.

Minimal partial realizations are in general not unique modulo isomorphism. Therefore, we introduce a natural partial realization, and we show that natural partial realizations exist if and only if they are isomorphic.

(7.23) Definition
For an Almost Linear System $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$ and some $p \in N$, if $X = \ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j)); \omega \in \Omega_p\} \gg$, then σ is said to be p-quasi-reachable.
Let q be some integer. If $hF^{q'}x = 0$ for any $q' \leq q$ implies $x = 0$, then σ is said to be q-observable.
For a given time-invariant, affine input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, if there exist p and $q \in N$ such that $p + q < \underline{N}$ and σ is p-quasi-reachable and q-observable then σ is said to be a natural partial realization of $\underline{a}$.

For a partial time-invariant, affine input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, the following matrix $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ is said to be a finite-sized Input/Output Matrix of $\underline{a}$.

$$\begin{array}{c} \\ (I/O)_{\underline{a}\ (p,\underline{N}-p)} = \\ t \end{array} \begin{array}{c} (s,u) \\ \left(\begin{array}{ccc} & \vdots & \\ & \vdots & \\ & \vdots & \\ \cdots \quad \cdots & a(u^{s+t+1}) - a(u^{s+t}) & \\ & & \end{array} \right) \end{array}$$

Where $0 \leq s \leq p$, $0 \leq t \leq \underline{N} - p$ and $u \in \{0, 1\}$.

(7.24) Theorem
Let $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ be the finite Input/Output Matrix of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then there exists a natural partial realization of $\underline{a}$ if and only if the following condition holds:
rank $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$= rank $(I/O)_{\underline{a}\ (p,\underline{N}-p-1)}$ =rank $(I/O)_{\underline{a}\ (p+1,\underline{N}-p-1)}$ for some $p \in N$.

[proof] See (7-C.9) in Appendix 7.7.C.

(7.25) Theorem
There exists a natural partial realization of a given partial time-invariant, affine input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ if and only if the minimal partial realizations of $\underline{a}$ are unique modulo isomorphism.

[proof] See (7-C.11) in Appendix 7.7.C.

(7.26) Theorem
Let a partial time-invariant, affine input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ satisfy the condition of Theorem (7.24), then the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ which realizes $\underline{a}$ can be obtained by the following algorithm:
Set $n :=$ rank $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$, where $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ is the finite Input/Output Matrix of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$.
1) Select the linearly independent vectors $\{S_l^{I_i}(\chi^0 + \bar{\chi}^{J_i}; J_i \in \{0,1\}, 1 \leq i \leq n\}$ of the set from $(I/O)_{a\ (p,\underline{N}-p)}$ in order of the numerical value.
2) Let the state space be K^n. Let the map $g_s^0 = \mathbf{e_1}$ and $g_s := \mathbf{e_2} - \mathbf{e_1}$. Where $J_2 = 0$. If $J_2 \neq 0$, $g_s := \alpha\mathbf{e_1}, \alpha \in K$.
3) Let the output map $h_s = [\underline{a}(J_1) - \underline{a}(\mathit{1}), \underline{a}(J_2^{I_2+1}) - \underline{a}(J_2^{I_2}), \cdots, \underline{a}(J_n^{I_n+1}) - \underline{a}(J_n^{I_n})]$
4) Let f_i in $F_s := [f_1, f_2, \cdot, f_n] \in K^{n\times n}$ be $f_i := [f_{i,1} f_{i,2}, \cdots, f_{i,n}]$.
Where $\underline{S_l}^{I_i+1}(\chi^0 + \bar{\chi} \cdot J_i) = \sum_{j=1}^{n} f_{i,j} \underline{S_l}^{I_j}(\chi 0 + \bar{\chi} J_j)$, $f_{i,j} \in K$ in the sense of $F(N_{\underline{N}-p}, Y)$.
Where $\underline{S_l} : F(N_s, Y) \to F(N_{s-1}, Y); \underline{a} \mapsto \underline{S_l}\underline{a}[; t \mapsto a(t+1)]$ for some $s \in N$.

[proof] See (7-C.12) in Appendix 7.7.C.

7.5 Real-Time Partial Realization Theory of Almost Linear System

In general, it is known that non-linear systems can be only determined by multi-experiments. In fact, in Chapter 3, we gave a condition for a general unknown black-box to be determined with a single-experiment. This condition may be very hard for us to find. However, we can look for special single-experiments to pretend multi-experiments for any Almost Linear System. In this section, on the results of partial realization theory in section 7.7.C, we will discuss a single-experiment for Almost Linear Systems.

(7.27) Real time partial realization problem
Let a physical object (equivalently, $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ be a finite dimensional Almost Linear System. Then for given finite data $\{a(\bar{\omega}); \bar{\omega}$ is a finite length input $\}$, find an Almost Linear System $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ and an input $\bar{\omega} \in \Omega$ such that $a_\sigma(\bar{\omega}) = a(\bar{\omega})$ for any $\omega \in \Omega$.

(7.28) Definition
For finite dimensional Almost Linear System, if there exists a solution of the real time partial realization problem, then an input $\bar{\omega} \in \Omega$ of the solution is said to be a (real time partial) realization signal.

(7.29) Lemma
Let a given time-invariant, affine input response map $a \in F(\Omega, Y)$ have the behavior of an Almost Linear System whose state space is less than L dimensional. Then there exists an input of finite length $\bar{\omega} \in \Omega$ such that the following algorithm provides a finite Input/Output Matrix.
Where $p := max\{L_1, L_2\}$.

1) Find an integer L_1 such that row vectors $\{\underline{S_l}^i \chi^0 \in K^L; 0 \leq i \leq L_1 - 1\}$ are linearly independent and $\{\underline{S_l}^i \chi^0 \in K^L; 0 \leq i \leq L_1\}$ are linearly dependent. Namely, feed an input $\omega_1 := 0^{L_1+L+1}$ into the plant.
Where $\underline{S_l}^i \chi^0 = [a(0^{n+1}) - a(0^n), a(0^n) - a(0^{n-1}), \cdots, a(0^{L+i+1}) - a(0^{L+i})]^T$.
2) Find an integer L_2 such that row vectors $\{\underline{S_l}^i \chi^0, \underline{S_l}^i (\chi^0 + \bar{\chi}) \in K^L; 0 \leq i \leq L_j - 1, 1 \leq j \leq 2\}$ are linearly independent and $\{\underline{S_l}^i \cdot \chi^0\ , \underline{S_l}^i (\chi^0 + \bar{\chi} \cdot u \in K^L; 0 \leq i \leq L_j, 1 \leq j \leq 2\}$ are linearly dependent. Namely, feed a further input $\omega_2 := 0^{L_1+L-1}|1$ into the plant.
Let $\bar{\omega} = \omega_2 | \omega_1$.
Making the row vectors of a matrix from the row vectors $\{\underline{S_l}^i (\chi^0 + \bar{\chi} \cdot u)) \in K_L; 0 \leq i \leq L_j,\ 1 \leq j \leq 2, u \in \{0, 1\}\}$ obtained by the above iterations, we will obtain a finite Input/Output Matrix $(I/O)_{a\ (L-1,p)}$.
Where $\underline{S_l}^i \bar{\chi} = [a(0^i|1) - a(0^{i+1}), a(0^{i+1}|1) - a(0^{i+2}), \cdots, a(0^{i+L}|1) - a(0^{i+L+1})]^T$.
And $a(0^i|1)$ is given by $a(0^j|1) = a(0^{i+1}|1|0^t) - a(0^{t+1}) + a(0^t)$ for any $i, t \in N$.

[proof] See (7-D.1) in Appendix 7.7.D.

(7.30) Theorem
Let a given time-invariant, affine input response map $a \in F(\Omega, Y)$ have the behavior of an Almost Linear System whose state space is less than L dimensional. Then there exists a realization signal such that the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ which realizes a can be obtained by the following algorithm:
1) Find a finite Input/Output Matrix $(I/O)_{a\ (L-1,p)}$ upon the algorithm given in Lemma (7.29).
2) Apply the algorithm given in Theorem (7.26) to the above finite Input/Output Matrix $(I/O)_{a\ (L-1,p)}$.

[proof] This can be obtained by Lemma (7.29).

7.6 Historical Notes and Concluding Remarks

We introduced Almost Linear Systems that are in a subclass of Pseudo Linear Systems, which are very close to linear systems. The reason depends upon the fact that there are So-called Linear Systems as examples of Almost Linear Systems. The So-called Linear Systems mean the linear systems with non-zero initial state. This means that the realization problem of Almost Linear Systems contains the realization problem of linear systems and the state estimation problem of linear systems simultaneously. For example, if we resolve the partial realization problem of Almost Linear Systems, then we naturally obtain both solutions of the partial realization problem and a state estimation problem for linear systems with an unknown initial state. Now, we cannot find such dynamical systems in other references.

First we showed that any time-invariant, affine input response map (equivalently, any input/output map with causality, time-invariance and affinity) can be completely characterized by a modified impulse response, which may be a slightly revised version of an impulse response in linear systems. We also showed that any Almost Linear Systems can be characterized by time-invariant, affine input response maps. This means that So-called Linear Systems can be completely characterized by modified impulse responses. Note that linear systems are completely characterized by impulse responses.

The set $A(N \times \{0,1\}, K)$ in Example (7.5) is new. Therefore, Nerode equivalence for $A(N \times \{0,1\}, K)$ in Theorem (7.6) is new. See Section 3.4 in Chapter 3 for the comments for nerode equivalence. It is shown that the uniqueness Theorem (7.8) holds in the sense of Almost Linear Systems; namely the theorem is stronger than in the sense of Pseudo Linear Systems;

Theorem (7.13) is one for finite dimensional Almost Linear Systems to be canonical. It can be easily understood that this theorem is an extension of the theorem for finite dimensional linear systems to be canonical. Also we gave the quasi-reachable standard system and the observable standard system that correspond to companion forms of linear systems. We gave a criterion for the behavior of finite dimensional Almost Linear Systems. The condition is given by finite rank of Input/Output Matrix, which is a natural extension of finite rank of Hankel matrix in linear systems. We gave a realization procedure to obtain the quasi-reachable standard system from a given time-invariant, affine input response map.

Moreover, we obtained partial Realization Theorems (7.24) and (7.25). Also we obtained a partial realization algorithm in Theorem (7.26). We discussed the real-time partial realization problem without any restriction. It depends upon time-invariance of input/output map.

7.7 Appendix

This Appendix 7.7 is prepared for the proof of the results about Almost Linear Systems.

7.7.A Realization Theory

In this section we will prove the main Theorem (7.8). To prove it, we equivalently convert the Almost Linear Systems to sophisticated Almost Linear Systems owing to results obtained in Appendix 7.7.A to 7.7.C. In Appendix 7.7.E we prove the realization theorem in the sophisticated Almost Linear Systems. This implies that Theorem (7.8) is proved.

7.7.A.1 Derivation Almost Linear Systems from Pseudo Linear Systems

We will show how Almost Linear Systems can be derived from Pseudo Linear Systems. For detail on the Pseudo Linear Systems, see Chapter 6.

(7-A.1) Lemma
If the behavior a_σ of an observable Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ satisfies the following affinity condition, then a map $g : U \to X$ is an affine map, i.e. $g(u) = g(0) + g(1) \cdot u$ holds for any $u \in U$.
Affinity condition:
$a_\sigma : \Omega \to Y$ is an affine map, i.e.,
$a_\sigma(\omega + \bar{\omega}) + a_\sigma(0^{|\omega|}) = a_\sigma(\omega) + a_\sigma(\bar{\omega})$
$a_\sigma(\lambda\omega) = \lambda a(\omega) + (1 - \lambda)a_\sigma(0^{|\omega|})$
for any $\omega, \bar{\omega} \in \Omega$, $|\omega| = |\bar{\omega}|$ and $\lambda \in K$.

[proof] This is obtained by direct calculation.

Remark 1: A Pseudo Linear System $\sigma = ((X, F), g, h, h^0)$ represents the following equations:

$$\begin{cases} x(t+1) & = & Fx(t) + g(\omega(t+1)) \\ x(0) & = & 0 \\ \gamma(t) & = & h^0 + hx(t) \end{cases}$$

for any $t \in N$, $x(t) \in X$, $\omega(t) \in U$ and $\gamma(t) \in Y$. And $F \in L(X)$, g is a map $: U \to X$.

(7-A.2) Definition
A Pseudo Linear System $\sigma = ((X,F),g,h,h^0)$ given by Lemma (7-A.1) is said to be an Almost Linear System. Therefore, the Almost Linear System $\sigma = ((X,F),g,h,h^0)$ represents the following equations:

$$\begin{cases} x(t+1) &= Fx(t) + g(0) + g(\omega(t+1)) \\ x(0) &= 0 \\ \gamma(t) &= h^0 + hx(t) \end{cases}$$

for any $t \in N$, $x(t) \in X$, $\omega(t) \in U$ and $\gamma(t) \in Y$. And $F \in L(X)$, $g(0) \in X$, g is a linear operator accompanied with $g : U \to X$.

(7-A.3) Lemma
For any linear operator $g : U \to X$, there uniquely exists $\bar{g} \in X$ such that $\bar{g} \cdot u = g(u)$ for any $u \in U$. This correspondence is bijective.

[proof] This can be obtained easily.

Remark 1: Lemma (7-A.3) implies that an Almost Linear System $\sigma = ((X,F),g,h,h^0)$ may be represented by the following equations:

$$\begin{cases} x(t+1) &= Fx(t) + g^0 + \bar{g}\omega(t+1) \\ x(0) &= 0 \\ \gamma(t) &= h^0 + hx(t) \end{cases}$$

Therefore, the Almost Linear System σ may be written by $\sigma = ((X,F),g^0,\bar{g},h,h^0)$.

for any $t \in N$, $x(t) \in X$, $\omega(t) \in U$ and $\gamma(t) \in Y$. And $F \in L(X)$, $g^0 = g(0), \bar{g} = g(1) \in X$.

Remark 2: This definition for an Almost Linear System is the same as Definition (7.4).

7.7.A.2 Free Motions with an Affine Map

In this section we will discuss free motions with an affine map needed for details of Almost Linear Systems, which are state structure with an input mechanism of Almost Linear Systems. We have already introduced free motions which are state structures of Pseudo Linear Systems. Note that the free motions are equivalent to state structures of Almost Linear Systems. For details of them, see section 6.7.B in Appendix 6.7. In that section, we have introduced free motions with an affine map as the input mechanism of Pseudo Linear Systems and free motions with an input map. Then we have shown that they are equivalent. In this section we will introduce free motions with an affine map and free motions with an input map which are more suitable for Almost Linear Systems. Specifically, we will clarify differences between Almost Linear Systems and Pseudo Linear Systems. We will show that free motions with an affine map and free motions with an input map are equivalent. Moreover, we will discuss quasi-reachability of free motions with an affine map.

(7-A.4) Definition
A system given by the following equation is written as a pair (X, F) and it is said to be a free motion.
$x(t+1) = Fx(t)$
Where X is a linear space over the field K and a linear map $F : X \to X$.
Let (X_1, F_1) and (X_2, F_2) be free motions, then a linear map $T : X_1 \to X_2$ is said to be a free motion morphism : $(X_1, F_1) \to (X_2, F_2)$ if T satisfies $TF_1 = F_2T$.

(7-A.5) Example
Let $A(N \times \{0,1\}, K)$ and S_r be the same as that considered in Example (7.5). Then $(A(N \times \{0,1\}, K), S_r)$ is a free motion.

(7-A.6) Example
In the set $F(N, Y)$ of any map from N to Y, let S_l be the same as in Example (7.5). Then $(F(N, Y), S_l)$ is a free motion.

(7-A.7) Definition
For free motions $(A(N \times \{0,1\}, K), S_r)$ and $(F(N, Y), S_l)$ considered in Examples (7-A.5) and (7-A.6), a free motion morphism:
$A : (A(N \times \{0,1\}, K), S_r) \to (F(N, Y), S_l)$ is said to be a linear input/output map.
For a free motion (X, F), a free motion morphism $G : (A(N \times \{0,1\}, K), S_r) \to (X, F)$ is said to be a linear input map, and a free motion morphism $H : (X, F) \to (F(N, Y), S_l)$ is said to be a linear observation map.

(7-A.8) Corollary
Any free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ can be normally decomposed into $X_1 \overset{\pi}{\to} X_1/\ker T \overset{T^b}{\to} \mathrm{im}\, T \overset{j}{\to} X_2$, where π is the canonical surjection, T^b is the isomorphism associated with T, j is the canonical injection and they are free motion morphisms respectively.

(7-A.9) Lemma
$(A(N \times \{0,1\}, K))^* = TF(\Omega, Y)$.
Where $(A(N \times \{0,1\}, K))^*$ is a set of any linear map from $A(N \times \{0,1\}, K)$ to Y, and $TF(\Omega, Y) := \{a \in F(\Omega, Y)$; a is a time invariant, affine input response map$\}$.

[proof] For any $a \in TF(\Omega, Y)$, set $\tilde{} : a \mapsto \tilde{a}[; \sum_{(n,u)} \lambda(n,u)\mathbf{e_{(n,u)}} \mapsto \sum_{(n,u)} \lambda(n,u)(a(u^{t+1}) - a(u^t))]$, then $\tilde{a} \in A(N \times \{0,1\}, K)^*$ holds. For any $\tilde{a} \in A(N \times \{0,1\}, K)^*$, set $e^* : \tilde{a} \mapsto \tilde{a} \cdot e^*[; \omega \mapsto \tilde{a}(\mathbf{e_{(n,\omega(1))}})$, then $\tilde{a} \cdot e^* \in TF(\Omega, Y)$ holds. Here, $e^* \cdot \tilde{} = I$ and $\tilde{} \cdot e^* = I$ hold. Hence $TF(\Omega, Y)$ is a concrete expression of $A(N \times \{0,1\}, K)^*$, we obtain $A(N \times \{0,1\}, K)^* = TF(\Omega, Y)$.

(7-A.10) Definition
For a free motion (X, F) and an affine map $g : U \to X$, i.e., $g(u) = g^0 + \bar{g} \cdot u$ for any $u \in U$, a collection $((X, F), g^0, \bar{g})$ is said to be a free motion with an affine map.
Where $g^0 := g(0)$.
A free motion with an affine map $((X, F), g^0, \bar{g})$ represents the following equations:
$x(t + 1) = Fx(t) + g^0 + \bar{g} \cdot \omega(t + 1)$.
Where $x(t), g^0, \bar{g} \in X$, $\omega(t) \in U$.
For the reachable set $\{\sum_{j=1}^{|\omega|}(g^0 + \bar{g} \cdot \omega(j)); \omega \in \Omega\}$, the smallest linear space which contains it is equal to X, then $((X, F), g^0, g)$ is said to be quasi-reachable.

(7-A.11) Example
For the free motion $(A(N \times \{0,1\}, K), S_r)$ considered in Example (7-A.5), $e_{(0,0)}$ and $\bar{\eta} := e_{(0,1)} - e_{(0,0)} \in A(N \times \{0,1\}, K)$, $((A(N \times \{0,1\}, K), S_r), e_{(0,0)}, \bar{\eta})$ is a free motion with an affine map and quasi-reachable.

(7-A.12) Example

For the free motion $(F(N,Y),S_l)$ considered in Example (7-A.6) and a time invariant, affine input response map $a \in F(\Omega,Y)$, let χ^0 and $\bar{\chi} \in F(N,Y)$ be $\chi^0(t) := a(\omega|0) - a(\omega)$, $\bar{\chi}(t) := a(\omega|1) - a(\omega|0)$.
Where $\omega \in \Omega$ and $|\omega| = t$ for $t \in N$. Then $((F(N,Y),S_l),\chi^0,\bar{\chi})$ is a free motion with an affine map.

(7-A.13) Definition

For free motions with an affine map $((X_1,F_1),g_1^0,\bar{g}_1)$ and $((X_2,F_2),g_2^0,\bar{g}_2)$, a free motion morphism $T : (X_1,F_1) \to (X_2,F_2)$ which satisfies $Tg_1^0 = g_2^0$ and $T\bar{g}_1 = \bar{g}_2$ is said to be a free motion morphism with an affine map $T : ((X_1,F_1),g_1^0,\bar{g}^1) \to ((X_2,F_2),g_2^0,\bar{g}^2)$.

(7-A.14) Proposition

For any free motion with an affine map $((X,F),g^0,\bar{g})$, there exists a unique free motion morphism with an affine map $G : ((A(N\times\{0,1\},K),S_r),e_{(0,0)},\bar{\eta}) \to ((X,F),g^0,g)$, where $G(e_{(0,0)}) = g^0$ and $G(\bar{\eta}) = \bar{g}$.
Conversely, for any free motion morphism $G : (A(N \times \{0,1\},K),S_r) \to (X,F)$, $((X,F),g^0,\bar{g})$ given by $g^0 = G(e_{(0,0)})$ and $\bar{g} = G(\bar{\eta})$ is a free motion with an affine map.

[proof] Let $G(e_{(0,0)}) = g^0$ and $G(\bar{\eta}) = \bar{g}$. Then G can be defined on $\{e_{(0,0)},e_{(0,1)}\}$. Since $GS_r = FG$, G can be extended to $\{e_{(t,u)}; t \in N, u \in \{0,1\}\}$. $\{e_{(t,u)}; t \in N, u \in \{0,1\}\}$ being the basis in $A(N \times \{0,1\},K)$, G is unique. The latter part is obvious.

Remark 1: According to Proposition (7-A.14), a linear input map $G : (A(N\times\{0,1\},K),S_r) \to (X,F)$ corresponds to an affine map $g : U \to X : u \mapsto g^0 + \bar{g} \cdot u$ is determined uniquely, and this correspondence is isomorphism.
Remark 2: If a free motion with an affine map $((X,F),g^0,\bar{g})$ in Proposition (7-A.14) is replaced with $((F(N,Y),S_l),\chi^0,\bar{\chi})$ considered in Example (7-A.6), then a linear input/output map $A : (A(N \times \{0,1\},K),S_r) \to (F(N,Y),S_l)$ corresponds to a time-invariant, affine input response map $a \in F(\Omega,Y)$ uniquely, and this correspondence is isomorphism.

(7-A.15) Corollary

For any linear input/output map $A : (A(N\times\{0,1\},K),S_r) \to (F(N,Y),S_l)$, there uniquely corresponds a time-invariant, affine input response map $a \in$

$F(\Omega, Y)$ such that $a(\omega|u) - a(\omega) = A(e_{(0,u)})(t)$ for any $t \in N$, $\omega \in \Omega$, $|\Omega| = t$ and $u \in \{0, 1\}$. This correspondence is isomorphism.

[proof] If a free motion with an affine map $((X, F), g^0, \bar{g})$ in Proposition (7-A.14) is replaced with $((F(N, Y), S_l), \chi^0, \bar{\chi})$, this corollary is obtained.

Remark: For a time-invariant, affine input response map $a \in F(\Omega, Y)$, there uniquely exists a linear input/output map $A : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$. On the other hand, for a time-invariant input response map $a \in F(\Omega, Y)$, there uniquely exists a linear input/output map $A : (A(N \times U, K), S_r) \to (F(N, Y), S_l)$.
Therefore, $A(N \times U, K)$ and $A(N \times \{0, 1\}, K)$ are different from a time -invariant input response map and a time-invariant, affine input response map.

(7-A.16) Proposition
A free motion with an affine map $((X, F), g^0, \bar{g})$ is quasi-reachable if and only if the corresponding linear input map G is surjective.

[proof] The quasi-reachability of $((X, F), g^0, \bar{g})$ means that the liner hull of the reachable set $X_1 := \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g} \cdot \omega(j)); \omega \in \Omega\}$ is equal to X. On the other hand, the surjection of G means that $X_2 := \{G(\lambda) = \sum_{(n,u)} \lambda(n, u) F^n(g^0 + \bar{g} \cdot u); \lambda = \sum_{(n,u)} e_{(n,u)} \in A(N \times \{0, 1\}, K)\}$ is equal to X. Here the equations imply that linear hull of X_1 is equal to X_2. Therefore, this proposition is obtained.

7.7.A.3 Free Motions with a Readout Map

In this section we will discuss free motions with a readout map. It is evident from Definitions (7.4) and (6.4) and the free motions with a readout map in Appendix 6.7 that state structures with a readout map in Almost Linear Systems are the same as state structures with a readout map in Pseudo Linear Systems. Hence, for readability, we will only list them.

(7-A.17) Definition
For a free motion (X, F) and a linear map $h : X \to Y$, a collection $((X, F), h)$ is said to be a free motion with a readout map. A free motion with a readout map $((X, F), h)$ represents the following equations:

$$\begin{cases} x(t+1) & = & Fx(t) \\ \gamma(t) & = & hx(t) \end{cases}$$

for any $t \in N$,where $x(t) \in X$ and $\gamma(t) \in Y$.
If $hF^n x_1 = hF^n x_2$ for any $n \in N$ implies $x_1 = x_2$, then $((X, F), h)$ is called observable.
Let $((X_1, F_1), h_1)$ and $((X_2, F_2), h_2)$ be free motions with a readout map, then a free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $h_1 = h_2 T$ is said to be a free motion morphism with a readout map $T : ((X_1, F_1), h_1)$
$\to ((X_2, F_2), h_2)$.

(7-A.18) Example
For the free motion $(A(N \times \{0, 1\}, K), S_r)$ considered in (7-A.5) and any time-invariant, affine input response map $a \in F(\Omega, Y)$, $((A(N \times \{0, 1\}, K), S_r)$,
$\bar{a})$ is a free motion with a readout map (see Lemma (7-A.9)).
Where a linear map $\bar{a} : A(N \times \{0, 1\}, K) \to Y$ is given by $\bar{a}(e_{(n,u)}) = a(u^{n+1}) - a(u^n)$ for any $n \in N$ and $u \in \{0, 1\}$.

(7-A.19) Example
Regarding the free motion $(F(N, Y), S_l)$ considered in Example (7-A.6), by defining a linear map $0 : (F(N, Y) \to Y; \gamma \mapsto \gamma(0)$, $((F(N, Y), S_l), 0)$ is a free motion with a readout map and it is observable.

(7-A.20) Proposition
For any free motion with a readout map $((X, F), h)$, there exists a unique linear observation map $H : (X, F) \to (F(N, Y), S_l)$ which satisfies $h = 0 \cdot H$, where $(Hx)(t) = hF^t x$ holds for any $x \in X$, $t \in N$.

Remark 1: According to Proposition (7-A.20), a linear observation map $H : (X, F) \to (F(N, Y), S_l)$ uniquely corresponds to a linear map $h : X \to Y$, and this correspondence is isomorphism.
Remark 2: If $((X, F), h)$ in Proposition (7-A.20) is replaced with $((F(N, Y), S_l), 0)$ considered in (7-A.6), a linear observation map $: (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$ is a linear input/output map. (See Corollary (7-A.15))

(7-A.21) Proposition
A free motion with a readout map $((X, F), h)$ is observable if and only if

the corresponding linear observation map $H : (X, F) \to (F(N, Y), S_l)$ is injective.

7.7.A.4 Almost Linear Systems

In this section we introduce sophisticated Almost Linear Systems, and show that Almost Linear Systems (said to be naive Almost Linear Systems) introduced in Definition (7.4) and sophisticated Almost Linear Systems are considered the same thing.

(7-A.22) Definition
A collection $\sigma = ((X, F), G, H, h^0)$ is said to be a sophisticated Pseudo Linear System if G is a linear input map : $(A(N \times \{0, 1\}, K), S_r) \to (X, F)$ and H is a linear observation map : $(X, F) \to (F(N, Y), S_l)$.
A linear input/output map $A_\Sigma = H \cdot G : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$ is said to be the behavior of Σ.
For a linear input/output map A and some $a(1) \in Y$, if $A_\Sigma = A$ and $h^0 = a(1)$, then a sophisticated Almost Linear System Σ is called a realization of $(A, a(1))$.
A sophisticated Pseudo Linear System $\Sigma = ((X, F), G, H, h^0)$ is called canonical if G is surjective and H is injective.
For $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$, a free motion morphism $T : (X_1, F_1) \to (X_2, F_2)$ which satisfies $TG_1 = G_2$ and $H_1 = H_2 T$ is said to be a sophisticated Almost Linear System morphism $: \Sigma_1 \to \Sigma_2$.
If T is surjective and injective then $T : \Sigma_1 \to \Sigma_2$ is said to be an isomorphism.

(7-A.23) Example
For the free motion $(A(N \times \{0, 1\}, K), S_r)$ in Example (7-A.5), identity map I on $A(N \times \{0, 1\}, K)$ and a linear input/output map $A : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$, a collection $((A(N \times \{0, 1\}, K), S_r), I, A, a(1))$ is a sophisticated Almost Linear System with the behavior $(A, a(1))$.
For the free motion $(F(N, Y), S_l)$ in Example (7-A.6), a linear input/output map A and identity map I on $F(N, Y)$, then a collection $((F(N, Y), S_l), A, I, a(1))$ is a sophisticated Almost Linear System with the behavior $(A, a(1))$.

In this situation, we consider the relation between sophisticated Almost Linear Systems and naive ones.

(7-A.24) Proposition
For any sophisticated Almost Linear System $\Sigma = ((X, F), G, H, a(1))$, there exists a unique naive Almost Linear System $\sigma = ((X, F), g^0, \bar{g}, h, a(1))$ corresponding to the sophisticated Almost Linear System Σ through the two equations (a.1) and (a.2).

$$G(\omega) = \sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j) \text{ for any } \omega \in \Omega \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a.1)$$

$$Hx(t) = hF^t x \text{ for any } x \in X \text{ and } t \in N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a.2)$$

This correspondence is isomorphic in the category's sense (see Pareigis).

[proof] It is easily obtained from Remark 2 of Proposition (7-A.14) and Remark 2 of Proposition (7-A.20).

7.7.A.5 Sophisticated Almost Linear Systems

In this section we will prove Realization Theorem (7.8).
According to Remark 2 in Proposition (7-A.14) (or Remark 2 in Proposition (7-A.20)) and Proposition (7-A.24), the realization theorem can be replaced with the following Theorem (7-A.25). Hence proving this theorem implies proving Realization Theorem (7.8).

(7-A.25) (Sophisticated) Realization Theorem
For any linear input/output map $A : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$ and some $a(1) \in Y$, there exist at least two sophisticated canonical Almost Linear Systems which realize $(A, a(1))$ (existence part).
Let $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$ be sophisticated canonical Almost Linear Systems which have the same behavior, then there exists an isomorphism $T : \Sigma_1 \to \Sigma_2$ (uniqueness part).

[proof] The next Corollary (7-A.26) signifies proving the existence part. Remark in Corollary (7-A.30) signifies proving the uniqueness.

(7-A.26) Corollary
For any linear input/output map $A : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$ and some $a(1) \in Y$, the following sophisticated Almost Linear Systems (1) and (2) are canonical realizations of $(A, a(1))$.

(1) $\Sigma_q = ((A(N \times \{0,1\}, K)/\ker A, \tilde{S}_r), \pi, A^i, a(1))$.
Where π is the canonical surjection : $A(N \times \{0,1\}, K) \to A(N \times \{0,1\}, K)/\ker A$ and A^i is given by $A^i = jA^b$ for $A^b : A(N \times \{0,1\}, K)/\ker A \to \mathrm{im}\ A$ being isomorphic with A and j being the canonical injection : $\mathrm{im}\ A \to F(N, Y)$.
(2) $\Sigma_s = ((\mathrm{im}\ A, S_l), A^s, j, a(1))$.
Where $A^s = A^b j$.

[proof] This can be obtained easily by Corollary (7-A.8), Example (7-A.23), the definition of canonicality and behavior.

Next, to prove the uniqueness part of Theorem (7-A.25), we introduce the following morphism $Mor(\Sigma_1, \Sigma_2)$ from a sophisticated Almost Linear System Σ_1 to another sophisticated Almost Linear System Σ_2.
Where Σ_1 and Σ_2 are given by $\Sigma_1 = ((X_1, F_1), G_1, H_1, h^0)$ and $\Sigma_2 = ((X_2, F_2), G_2, H_2, h^0)$.
$Mor(\Sigma_1, \Sigma_2) := \{$ relation $T : X_1 \to X_2; GrT_{12}^{min} \subseteq GrT_{12} \subseteq GrT_{12}^{max}\}$.
Where GrT_{12}^{min}, GrT_{12} and GrT_{12}^{max} denote the graph of $T_{12}^{min} := G_2 \cdot G_1^{-1}$, T_{12} and $T_{12}^{max} := H_2^{-1} H_1$ respectively.

Why this morphism is introduced depends on the next lemma.

(7-A.27) Lemma
$A_{\Sigma_1} = A_{\Sigma_2}$ if and only if $Mor(\Sigma_1, \Sigma_2) \neq \emptyset$.

[proof] This can be proved the same as in Matsuo [1981].

(7-A.28) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold.
(1) If G_1 of Σ_1 is surjective, then dom $T_{12}^{min} = X_1$ holds, where dom T_{12}^{min} denotes the domain of T_{12}^{min}.
(2) If H_2 of Σ_2 is injective, then T_{12}^{max} is a partial function : $X_1 \to X_2$.

[proof] This can be proved the same as lemma 4 in Matsuo [1981].

(7-A.29) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, then GrT_{12}^{max} is an invariant sub-product linear U-action of (X_1, F_1) and (X_2, F_2).

[proof] By the definition of GrT_{12}^{max}, $GrT_{12}^{max} = \{(x_1, x_2) \in X_1 \times X_2; H_1x_1 = H_2x_2\}$ holds. Let (x_1, x_2) and $(x_{1'}, x_{2'}) \in GrT_{12}^{max}$, i.e., $H_1x_1 = H_2x_2$ and $H_1x_{1'} = H_2x_{2'}$ hold. $H_1(x_1 + x_{1'}) = H_1x_1 + H_1x_{1'} = H_2x_2 + H_2x_{2'} = H_2(x_2 + x_{2'})$ hold. This implies $(x_1 + x_{1'}, x_2 + x_{2'}) \in GrT_{12}^{max}$. For $k \in K$ and $(x_1, x_2) \in GrT_{12}^{max}$, $(kx_1, kx_2) \in GrT_{12}^{max}$ holds. Moreover, for $(x_1, x_2) \in GrT_{12}^{max}$, $H_1F_1x_1 = S_lH_1x_1 = S_lH_2x_2 = H_2F_2x_2$ hold. Hence, we obtain $(F_1x_1, F_2x_2) \in GrT_{12}^{max}$. Therefore, $GrT_{12}^{max} \subseteq X_1 \times X_2$ is invariant under $F_1 \times F_2$. Therefore, $(GrT_{12}^{max}, F_1 \times F_2)$ is a free motion.

(7-A.30) Lemma
Let $A_{\Sigma_1} = A_{\Sigma_2}$ hold, G_1 be surjective and H_2 be injective, then $T_{12}^{min} = T_{12}^{max}$ holds and T_{12} is an Almost Linear System morphism : $\Sigma_1 \to \Sigma_2$ by setting $T_{12} = T_{12}^{min}$.

[proof] If G_1 is surjective and H_2 is injective, then Lemma (7-A.28) implies that $T_{12} \in Mor(\Sigma_1, \Sigma_2)$ is unique, $T_{12}G_1 = G_2$ and $H_2T_{12} = H_1$ hold. Owing to Lemma (7-A.29), T_{12} is a free motion morphism : $(X_1, F_1) \to (X_2, F_2)$.

Remark: The uniqueness part of (sophisticated) Realization Theorem (7-A.25) for time-invariant, affine input response maps is proven by sophisticated Almost Linear Systems being canonical and Lemma (7-A.30).

7.7.B Finite Dimensionality

In this Appendix, we will give proofs for theorems, propositions and corollaries stated in section 7.3.

7.7.B.1 Finite Dimensional Free Motions with an Affine Map

In Appendix 7.7.A.2, the free motions with an affine map were introduced. In this section we consider those whose state space is finite dimensional. Then it is shown that finite dimensional free motions can be represented by matrix expressions.

(7-B.1) Definition
A free motion with an affine map $((X, F), g^0, \bar{g})$ whose X is finite (n) dimensional is said to be a finite dimensional (n dimensional) free motion with an affine map.

In Appendix 7.7.A, we showed that the initial object of any free motion with an affine map $((X,F),g^0,\bar{g})$ is $((A(N\times\{0,1\},K),S_r),e_{(0,0)},\bar{\eta})$ and the quasi-reachability of $((X,F),g^0,\bar{g})$ implies a surjection of the corresponding linear input map G.

In this section we will give a criterion for being quasi-reachable of finite dimensional free motions with an affine map. Introducing the quasi-reachable standard form, we show that it is a representative of free motions with an affine map. These results will be obtained from the fact that free motions with an affine map are different from ones in Pseudo Linear Systems. Therefore, the results will be listed. See Appendix 6.7.B for the results of ones in Pseudo Linear Systems.

Let $((X,F),g^0,\bar{g})$ be a free motion with an affine map and G be the linear input map corresponding to an affine map $g^0,\bar{g}$, namely, a free motion morphism $G:(A(N\times\{0,1\},K),S_r)\to(X,F)$ which satisfies $G(e_{(0,0)})=g^0, G(\bar{\eta})=\bar{g}$.

Let $LR(i)$ be the linear hull of reachable set by input whose length is within i, i.e., $LR(i):=\ll\{\sum_{j=1}^{|\omega|}F^{|\omega|-j}(g^0+\bar{g}\omega(j);\omega\in\Omega_{i+1}\}\gg$.
Where $\Omega_i:=\{\omega\in\Omega;|\omega|\le i\}$.
Then the following formula holds:

$LR(0)=\ll\{g^0,\bar{g}\}\gg$,
$LR(i+1)=LR(i)+\ll\{Fx+g^0+\bar{g}u;u\in U,x\in LR(i)\}\gg$.
Therefore, the following sequence can be obtained:
$LR(0)\subseteq LR(1)\subseteq\cdots\subseteq LR(i)\subseteq\cdots\subseteq LR(\infty)$.
And $LR(n)=G(A(N\times\{0,1\},K)_n)$ holds. Where $A(N\times\{0,1\},K)_n$ denotes $\{\sum_{q,u}\lambda(q,u)e_{(q,u)}\in A(N\times\{0,1\},K),\ q\le n$ for $n\in N\}$.
Moreover, let $G_l=G\cdot J_l$, where J_l is the canonical injection : $A(N\times\{0,1\},K)_l\to A(N\times\{0,1\},K)$. Then the above sequence can be rewritten as the following:
$\text{im } G_0\subseteq\text{im } G_1\subseteq\cdots\subseteq\text{im } G_i\subseteq\cdots\subseteq\text{im } G_\infty$.
Then we can obtain the next lemma easily.

(7-B.2) Lemma
If $\text{im } G_{j-1}=\text{im } G_j$ for an integer $j\in N$, then $\text{im } G_j=\text{im } G_{j+1}$.

(7-B.3) Lemma

For any free motion with an affine map $((K^n, F), g^0, \bar{g})$, then im $G_{n-1} =$ im G always holds. Therefore, $(($im $G_{n-1}, F), g^0, \bar{g})$ is a quasi-reachable free motion with an affine map.

(7-B.4) Proposition

Let $((K^n, F), g^0, \bar{g})$ be a free motion with an affine map, then $((K^n, F), g^0, \bar{g})$ is quasi-reachable if and only if im $G_{n-1} = K^n$ holds.

(7-B.5) Proposition

Let $((K^n, F), g^0, \bar{g})$ be a quasi-reachable free motion with an affine map, then im G_{j-1} is more than j dimensional for any integer $j(1 \leq j \leq n)$.

(7-B.6) Proposition

Let $((K^n, F), g^0, \bar{g})$ be a free motion with an affine map. $((K^n, F), g^0, \bar{g})$ is quasi-reachable if and only if
rank $[g^0, \bar{g}, Fg^0, F\bar{g}, \cdots, F^{n-1}g^0, F^{n-1}\bar{g}] = n$ holds.

(7-B.7) Definition

Let $((K^n, F_s), g_s^0, \bar{g}_s)$ be a quasi-reachable free motion with an affine map. If $((K^n, F_s), g_s^0, \bar{g}_s)$ satisfies the following conditions, then it is said to be the quasi-reachable standard form:

1) $e_i = F_s^{I_i}(g_s^0 + \bar{g}_s \cdot J_i)$ holds for a set $\{(I_i, J_i) \in N \times \{0,1\}, 1 \leq i \leq n\}$.

2) $(I_1, J_1) < (I_2, J_2) < \cdots < (I_n, J_n)$ holds.

3) $I_i < i$

4) $F_s^p(g_s^0 + \bar{g}_s u_q) = \sum_{i=1}^{j} \alpha_i e_i$ holds for any $(p, u_q) \in N \times \{0,1\}$ such that $(I_j, J_j) < (p, u_q) < (I_{j+1}, J_{j+1})$.

Where $\alpha_i \in K$ and $e_i = [0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T$.

For the order by the numerical value, see Definition (7.14).

(7-B.8) Proposition

For any quasi-reachable free motion with an affine map $((K^n, F), g^0, \bar{g})$, there uniquely exists the quasi-reachable standard form $((K^n, F_s), g_s^0, \bar{g}_s)$, which is isomorphic to it.

[proof] We select the set of linearly n independent vectors $\{F^{I_i}(g^0 + \bar{g}J_i); (I_i, J_i) \in N \times \{0,1\}, 1 \leq i \leq n\}$ among $\{F^i(g^0 + \bar{g}u_j); i \in N, u_j \in \{0,1\}\}$ in the order of numerical value of $N \times \{0,1\}$. Then the condition $I_i < i$ for $i(1 \leq i \leq n)$ holds by Proposition (7-B.5). We introduce a linear opera-

tor $T : K^n \to K^n$ by setting $TF^{I_i}(g^0 + \bar{g}J_i) = e_i$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_s := TFT^{-1}$ and $g_s^0 := Tg^0$, $\bar{g}_s := T\bar{g}$, then $F_s \in K^{n \times n}$ and a collection $((K^n, F_s), g_s^0, \bar{g}_s)$ is a free motion with an affine map. Since $TF^{I_i}(g^0 + \bar{g}J_i) = e_i$ for $i(1 \leq i \leq n)$, the state e_i is a reachable state by a pair (I_i, J_i), $I_i < i - 1$. T is a free motion morphism with an affine map : $((K^n, F), g^0, \bar{g}) \to ((K^n, F_s), g_s^0, \bar{g}_s)$. T preserves the linear independence and dependence. Hence, $((K^n, F_s), g_s^0, \bar{g}_s)$ is a quasi-reachable standard form.
Moreover, we can show the uniqueness of it come from the selection of $\{(I_i, J_i) \in N \times \{0, 1\}, 1 \leq i \leq n\}$.

Remark: There are many equivalence classes in the category of free motions with an affine map, and this proposition says that the equivalence classes can be represented as quasi-reachable standard forms.

7.7.B.2 Finite Dimensional Free Motions with a Readout Map

In Appendix 7.7.A.3, we showed that a final object of any free motion with a readout map $((X, F), h)$ is $((F(N, Y), S_l, 0)$ and the distinguishability of $((X, F), h)$ implies an injection of the corresponding linear observation map H. In this section we will give a criterion for being observable of finite dimensional free motions with a readout map.

Introducing the observable standard form, we show that it is a representative of free motions with a readout map. Free motions with a readout map are the same as free motions with a readout map in Pseudo Linear Systems.

For readability, the results about finite dimensional free motions with a readout map will be listed.

Let $((X, F), h)$ be a free motion with a readout map and H be the linear observation map corresponding to a readout map h, namely, a free motion morphism $H : (X, F) \to (F(N, Y), S_l)$ which satisfies $0H = h$.
Let $LO(i)$ be the linear hull of reachable set by outputs whose length is within i, i.e., $LO(i) := \{\sum_j \lambda_j x'_j; \lambda_j \in K, x'_j \in X^*; x'_j = hF^j, 0 \leq j \leq i\}$. Then the following sequence holds.
$LO(0) \subseteq LO(1) \subseteq \cdots \subseteq LO(i) \subseteq \cdots \subseteq LO(\infty)$.

Let $H_l = P_l H$, where P_l is the canonical surjection : $F(N, Y) \to F(N_l, Y)$, where $F(N_l, Y) := \{a \in F(N, Y); a : N_l \to Y\}$ and $N_l := \{j \in N; j \leq l\}$.
Then ker $H_l = LO(l)^0$ holds; i.e., ker $H_l = \{x \in X; hx = 0 \text{ for } h \in LO(l)\}$.

Moreover, ker $H = LO(\infty)^0$ holds.

(7-B.9) Lemma
For any free motion with a readout map $((X, F), h)$, $LO(n-1) = \ll hF^N \gg$ holds.
Where $hF^N = \{hF^i; i \in N\}$.

(7-B.10) Proposition
For any free motion with a readout map $((X, F), h)$, ((ker $H_{n-1}, F)$ is a sub free motion of (K^n, F) and $((K^n/\text{ker } H_{n-1}, \tilde{F}), \tilde{h})$ is an observable free motion with a readout map.

(7-B.11) Proposition
Let $((X, F), h)$ be a free motion with a readout map. $((X, F), h)$ is observable if and only if $LO(n-1) = K^{p \times n}$ holds.

(7-B.12) Proposition
If $((X, F), h)$ is observable, then $LO(j-1)$ is more than j dimensional for any j $(1 \leq j \leq n)$.

(7-B.13) Proposition
Let $((X, F), h)$ be a free motion with a readout map. $((X, F), h)$ is observable if and only if
rank $[h^T, (hF)^T, (hF^2)^T, \cdots, (hF^{n-1})^T] = n$ holds.
Where T denotes the transpose of matrix.

(7-B.14) Definition
Conveniently, let Y be K. Let $((K^n, F_o), h_o)$ be an observable free motion with a readout map. If $((K^n, F_o), h_o)$ satisfies the following conditions, then it is said to be the observable standard form:
1) $e_i^T = h_o F_o^{i-1}$ holds for $i (1 \leq i \leq n)$.
2) $h_o F_o^n = \sum_{i=1}^n \alpha_i e_i^T$ holds.

Remark: If $((X, F), h)$ is the observable standard form, note that $h = e_1^T$.

(7-B.15) Propositon
For any observable free motion with a readout map $((X, F), h)$, there exists uniquely the observable standard form $((K^n, F_o), e_1^T)$ which is isomorphic to it.

Where Y is K.

[proof] We select the set of n linearly independent vectors $\{hF^{i-1}; 1 \leq i \leq n\}$. We introduce a linear operator $T : K^n \to K^n$ by setting $hF^{i-1} = e_i^T$ for $i(1 \leq i \leq n)$, then T is a regular matrix. Let $F_o := TFT^{-1}$, then $F_o \in K^{n \times n}$ and a collection $((K^n, F_o), e_1^T)$ is a free motion with a readout map. T is a free motion morphism with a readout : $((K^n, F), h) \to ((K^n, F_o), e_1^T)$. T preserves the linear independence and dependence. Hence $((K^n, F_o), e_1^T)$ is the observable standard form.

Remark: There are many equivalence classes in the category of free motions with a readout map, and this proposition says that the equivalence classes can be represented as the observable standard forms.
Conveniently, we assume that Y is K.

7.7.B.3 Finite Dimensional Almost Linear Systems

This section is prepared for the proofs of Representation Theorems (7.16) and (7.18) for finite dimensional canonical Almost Linear Systems.

(7-B.16) Proof of Representation Theorem (7.16)
Note that a free motion with an affine map of the quasi-reachable standard system is the quasi-reachable standard form.
Let $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ be any finite dimensional canonical Almost Linear System. For the quasi-reachable standard form $((K^n, F_s), g_s^0, \bar{g}_s)$ and a free motion morphism with an affine map:
$T : ((K^n, F), g^0, \bar{g}) \to ((K^n, F_s), g_s^0, \bar{g}_s)$ introduced in the proof of Proposition (7-B.8), let $h_s := h \cdot T^{-1}$. Then T is an Almost Linear System morphism : $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0) \to \sigma_s = ((K^n, F_s), g_s^0, \bar{g}_s, h_s, h^0)$. T is bijective and σ_s is the only quasi-reachable standard system. By Corollary (7.9), the behaviors of σ and σ_s are the same.

(7-B.17) Proof of Representation Theorem (7.18)
Note that a linear U-action with a readout map of the observable standard system is the observable standard form. Let $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ be any finite dimensional canonical Almost Linear System. For the observable standard form $((K^n, F_o), e_1^T)$ and a free motion morphism with a readout $T : ((K^n, F), h) \to ((K^n, F_o), e_1^T)$ introduced in the proof of Proposition

(7-B.15), let $g_o^0 := Tg_0$ and $\bar{g}_o := T\bar{g}$. Then T is an Almost Linear System morphism : $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0) \to \sigma_o = ((K^n, F_o), g_o^0, \bar{g}_o, e_1^T)$. T is bijective and σ_o is obviously the only observable standard system. By Corollary (7.9), the behaviors of σ and σ_o are the same.

7.7.B.4 Existence Criterion for Almost Linear Systems

This section is prepared for the proof of the theorem for existence criterion (7.20).
Let $G_l = G \cdot J_l$, where J_l is the canonical injection : $A(N \times \{0,1\}, K)_l \to A(N \times \{0,1\}, K)$. Let $H_l = P_l \cdot H$, where P_l is the canonical surjection : $F(N, Y) \to F(N_l, Y)$.

(7-B.18) Proof of Theorem (7.20)
Let A be the linear input/output map corresponding to a time-invariant, affine input response map $a \in F(\Omega, Y)$. Then a linear input/output map A is a free motion morphism $A : A(N \times \{0,1\}, K) \to (F(N, Y), S_l)$. On the other hand, for a time-invariant input response map $a \in F(\Omega, Y)$, a free motion morphism $A : A(N \times \{0,1\}, K) \to (F(N, Y), S_l)$ could be equivalently introduced in Chapter 6 and the theorem for existence criterion of Pseudo Linear Systems (6.19) could be obtained (see (6-B.18) in Appendix 6.7.B). If we compare time-invariant input response maps with time-invariant, affine input response maps, the difference is in $(A(N \times \{0,1\}, K), S_r)$ and $((A(N \times U, K), S_r)$. Namely, the difference comes from $\{0,1\}$ or U. Therefore, this theorem is obtained as a special case of Theorem (6.19) (see (6-B.18) in Appendix 6.7).

7.7.B.5 Realization Procedure for Almost Linear Systems

This section is prepared for the proof of theorem for Realization Procedure (7.21).

(7-B.19) Proof of Theorem (7.21)
Let $R(\chi) = \{S_l^i(\chi^0 + \bar{\chi} \cdot u); u \in \{0,1\}, i \in N\}$. By Theorem (7.6), $((\ll R(\chi) \gg, S_l), \chi^0, \bar{\chi}, 0, a(1))$ is a canonical Almost Linear System which realizes a time-invariant, affine input response map $a \in F(\Omega, Y)$. The linearly independent vectors $\{S_l^i(\chi^0 + \bar{\chi} \cdot u); u \in \{0,1\}, i \in N, 1 \le i \le n\}$ satisfies $\ll \{S_l^i(\chi^0 + \bar{\chi} \cdot J_i); i \in N, 1 \le i \le n, (I_1, J_1) < (I_2, J_2) < \cdots < (I_n, J_n)\} \gg = \ll R(\chi) \gg$. Let a linear map $T : \ll R(\chi) \gg \to K^n$ be $T \cdot$

$S_l^i(\chi^0 + \bar{\chi} \cdot J_i) = e_i$ for any $i(1 \leq i \leq n)$. Then, by step 2), $T\chi^0 = g_s^0$ and $T\bar{\chi} = g_s$ hold and by step 3), $h_s \cdot T = 0$ holds. By step 4), $F_s \cdot T = T \cdot S_l$ holds. Consequently, T is bijective and an Almost Linear System morphism : $((\ll R(\chi) \gg, S_l), \chi^0, \bar{\chi}, 0, a(1)) \to \sigma_s = ((K^n, F_s), g_s^0, \bar{g}_s, h_s, a(1))$. By Corollary (7.9), the behavior of σ_s is a. It follows from the choice of $\{S_l^i(\chi^0 + \bar{g} \cdot J_i), (I_1, J_1) < (I2, J2) < \cdots < (I_n, J_n)\}$ for $i \in N$, $1 \leq i \leq n$, $J_i \in \{0, 1\}\}$ and the determination of map T that σ_s is the quasi-reachable standard system.

7.7.C Partial Realization

In this Appendix, we give proofs for theorems and propositions stated in section 7.4. See Appendix 7.7.A and B for details of notions and notations.

7.7.C.1 Free Motions with an Affine Map

Set $A(N \times \{0,1\}, K)_p := \{\sum_{q,u} \lambda(q,u) e_{(q,u)} \in A(N \times \{0,1\}, K),\ q \leq p$ for $p \in N\}$.
J_p is the canonical injection : $A(N \times \{0,1\}, K)_p \to A(N \times \{0,1\}, K)$. Let $H_q = P_q \cdot H$, where P_q is the canonical surjection : $F(N, Y) \to F(N_q, Y)$.

(7-C.1) Definition
If a free motion with an affine map $((X, F), g^0, \bar{g})$ satisfies:
$X = \ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g} \cdot \omega(j)); \omega \in \Omega_p\} \gg$, then $((X, F), g^0, \bar{g})$ is said to be p-quasi reachable.

Remark: Note that $((X, F), g^0, \bar{g})$ is p-quasi reachable if and only if $G_p := G \cdot J_p : A(N \times \{0,1\}, K)_p \to X$ is surjective.
Where G is the linear input map $G : A(N \times \{0,1\}, K), S_r) \to (X, F)$ corresponding to $((X, F), g^0, \bar{g})$.

(7-C.2) Proposition
If a linear sub space S of $A(N \times \{0,1\}, K)_{p+1}$ satisfies the next two conditions, then there uniquely exists an ideal $\underline{S} \subset A(N \times \{0,1\}, K)$ such that $\underline{S} \cap A(N \times \{0,1\}, K)_{p+1} = S$ and $A(N \times \{0,1\}, K)_{p+1}/S$ is isomorphic to $A(N \times \{0,1\}, K)/\underline{S}$. Moreover, a free motion with an affine map $((A(N \times \{0,1\}, K)/\underline{S}, \tilde{S}_r), [e_{(0,0)}], [\bar{\eta}])$ is p-quasi-reachable.
Where $\tilde{S}_r$ is given by $\tilde{S}_r(\lambda + \underline{S}) = S_r\lambda + \underline{S}$ for $\lambda \in A(N \times \{0,1\}, K)$ and $[\bar{\eta}] = [e_{(0,0)} - e_{(0,1)}]$.

condition 1: $\lambda \in A(N \times \{0,1\}, K)_p \cap S$ implies $S_r\lambda \in S$.
condition 2: There exist coefficients $\lambda(q,u') \in K$ such that $e_{(p+1,u')} - \sum_{q,u'} \lambda(q,u')e_{(q,u')} \in S$ for $q \leq p$.

[proof] Let $U = \{0,1\}$ in Proposition (6-C.2) of Appendix 6.7. This proposition corresponds to it. Hence, this will be obtained.

7.7.C.2 Free Motions with a Readout Map

Set $F(N_q, Y) := \{\gamma \in F(N, Y)$; a function $\gamma : N_q \to Y\}$, let P_q be the canonical surjection : $F(N,Y) \to F(N_q,Y); \gamma \mapsto [; t \mapsto \gamma(t)]$, and define $\underline{S_l}$ by setting $\underline{S_l} : F(N_q,Y) \to F(N_{q-1},Y); \gamma \mapsto \underline{S_l}\gamma[; t \mapsto \gamma(t+1)]$.

(7-C.3) Definition
If a free motion with a readout map $((X,F),h)$ satisfies that $hF^t x = 0$ for any $t \in N_q$ implies $x = 0$, it is said to be q-observable.

Remark: Note that $((X,F),h)$ is q-observable if and only if a linear map $H_q := P_q \cdot H$ is injective. Where H is a linear output map corresponding to $((X,F),h)$.

(7-C.4) Proposition
If a sub space Z of $F(N_{q+1},Y)$ satisfies the next two conditions, then there exists uniquely a free motion (X, S_l) such that a map $P_{q|X} : X \to Z$ is isomorphic.
Where $P_{q|X}$ is a restriction of the canonical surjection $P_q : F(N,Y) \to F(N_q,Y)$ to X. A free motion with a readout map $((X,S_l),0)$ is q-observable.
condition 3: A composition map $\pi \cdot j : Z \xrightarrow{j} F(N_{q+1},Y) \xrightarrow{\pi} F(N_q,Y)$ is injective.
condition 4: im $(\underline{S_l} \cdot j) \subseteq$ im $(j \cdot \pi)$ holds in the sense of $F(N_q,Y)$.
Where π is the canonical surjection .

[proof] This is the same as Proposition (6-C.4) in Appendix 6.7.C.

7.7.C.3 Partial Realization Problem

We can consider a partial linear input/output map $A_{(p,\underline{N}-p)}$: $A(N \times \{0,1\}, K)_p \to F(N_{\underline{N}-p}, Y)$ for $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ the same as the linear input/output map $A : (A(N \times \{0,1\}, K), S_r) \to (F(N,Y), S_l)$ considered for $a \in F(\Omega, Y)$ in Appendix 7.7.A.

(7-C.5) Lemma

Let $A_{(p,\underline{N}-p)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then the following diagrams commute.

1)

$$\begin{array}{ccc} A(N \times \{0,1\}, K)_p & \xrightarrow{A_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p}, Y) \\ \Big\downarrow \underline{i} & & \Big\downarrow \pi \\ A(N \times \{0,1\}, K)_{p+1} & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1}, Y) \end{array}$$

Where $\underline{i}$ is a canonical injection and π is a canonical surjection.

2)

$$\begin{array}{ccc} A(N \times \{0,1\}, K)_p & \xrightarrow{A_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p}, Y) \\ \Big\downarrow S_r(u) & & \Big\downarrow \underline{S_l} \\ A(N \times \{0,1\}, K)_{p+1} & \xrightarrow{A_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1}, Y) \end{array}$$

[proof] These can be obtained by direct calculation.

(7-C.6) Proposition

Let $A_{(p_1,\underline{N}-p_1)}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and p_2 be any integers such that $0 \leq p_2 \leq p_1 < \underline{N}$.
If im $A_{(p_2+1,\underline{N}-p_2-1)} =$ im $A_{(p_2,\underline{N}-p_2-1)}$ then im $A_{(p_1,\underline{N}-p_1)} =$ im $A_{(p_2,\underline{N}-p_1)}$ holds.

[proof] Let $U = \{0, 1\}$ in Proposition (6-C.6) of Appendix 6.7.C. This proposition corresponds to it. Hence, this will be obtained.

(7-C.7) Proposition

Let $A_{(\ ,\ )}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Let p_1 and p_2 be any integers such that $0 \leq p_2 < p_1 < \underline{N}$.
If $\ker A_{(p_1, \underline{N}-p_1)} = \ker A_{(p_1, \underline{N}-p_1-1)}$ hold, then $\ker A_{(p_2, \underline{N}-p_1-1)}$ $= \ker A_{(p_2, \underline{N}-p_2)}$ holds.

[proof] This is the same as Proposition (6-C.7) in Appendix 6.7.C.

(7-C.8) Lemma

For a partial linear input/output map $A_{(\ ,\ )}$ corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ and an Almost Linear System $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$, the next matters hold.
Where $G_p := G \cdot J_p$, $H_q := P_q \cdot H$ for the linear input map G corresponding to $g^0, \bar{g}$ and the linear output map H corresponding to h. $A_{(p,q)} := H_q \cdot J_p$.
1) σ is a partial realization of $\underline{a}$ if and only if the following figure commutes for any p such that $0 \leq p < \underline{N}$.
2) σ is a natural partial realization of $\underline{a}$ if and only if the following figure commutes, G_p is surjective and $H_{\underline{N}-p-1}$ is injective for some p such that $0 \leq p < \underline{N}$.

$$\begin{array}{ccccc}
A(N \times \{0,1\}, K)_p & \xrightarrow{G_p} & X & \xrightarrow{H_{\underline{N}-p}} & F(N_{\underline{N}-p}, Y) \\
\downarrow{\scriptstyle S_r} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \underline{S_l}} \\
A(N \times \{0,1\}, K)_{p+1} & \xrightarrow{G_{p+1}} & X & \xrightarrow{H_{\underline{N}-p-1}} & F(N_{\underline{N}-p-1}, Y)
\end{array}$$

[proof] These can be obtained by definition of the partial and natural partial realization.

(7-C.9) Proof of Theorem (7.24)
We prove the theorem by rewriting the conditions of partial Input/Output Matrix in Theorem (7.24) to partial linear input/output map $A_{(\ , \ )}$ corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. By using Propositions (7-C.6) and (7-C.7), the conditions of Input/Output Matrix can be equivalently changed to the following equations (1) & (2):
(1) im $A_{(p,\underline{N}-p-1)} =$ im $A_{(p+1,\underline{N}-p-1)}$.
(2) ker $A_{(p,\underline{N}-p)} =$ ker $A_{(p,\underline{N}-p-1)}$.
Therefore, we will prove the theorem by using (1) and (2).
Firstly, we show that the above equations (1) & (2) are necessary. Let $\sigma = ((X,F), g^0, \bar{g}, h, h^0)$ be a natural partial realization of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, then σ is p-quasi-reachable and q-observable for some p and q such that $p + q < \underline{N}$.
Let G be the linear input map corresponding to $g^0, \bar{g}$ and H be the linear observation map corresponding to h, and let $p \le p'$ and $q \le q'$, then $G_{p'} := G \cdot J_{p'}$ is onto, $H_{q'} := P_{q'} \cdot H$ is one-to-one. Therefore, $A_{(p',q')} := H_{q'} \cdot J_{p'}$ satisfies conditions 1 and condition 2 in Proposition (7-A.2).
Next, we show that the equations (1) & (2) are sufficient.
Set $S :=$ ker $A_{(p+1,\underline{N}-p-1)}$ and $Z :=$ im $A_{(p,\underline{N}-p)}$. Then equation (2) implies that a composition map $\pi \cdot j : Z \xrightarrow{j} F(N_{\underline{N}-p}, Y) \xrightarrow{\pi} F(N_{\underline{N}-p-1}, Y)$ is injective. Where π and j are the same as in Proposition (7-C.4). Hence Z satisfies condition 3 in the Proposition. Equation (1) implies that there exist $e_{(l,u_i)} \in A(N \times \{0,1\}, K)_p$ such that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)}) = A_{(p,\underline{N}-p-1)}(\sum_i \lambda(l,u_i) \times e_{(l,u_i)})$ for any $u \in \{0,1\}$.
By Lemma (7-C.5), we obtain that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)} - \sum_i \lambda(l,u_i) e_{(l,u_i)}) = 0$, and $e_{(p+1,u)} - \sum_i \lambda(l,u_i) e_{(l,u_i)} \in S$ holds.

This implies that S satisfies condition 2 in Proposition (7-C.2). Let j be the canonical injection : $A_{(p,\underline{N}-p-1)} \to F(N_{\underline{N}-p-1}, Y)$ and π be the same as in Proposition (7-C.4), $B := (j)^{-1} \cdot \pi \cdot j : Z \to$ im $A_{(p,\underline{N}-p-1)}$ is a bijective linear map by (2) in Proposition (7-C.2). When we consider the bijective linear map $A^b := A^b_{(p+1,\underline{N}-p-1)} : A(N \times \{0,1\}, K)_{p+1}/S \to$ im $A_{(p+1,\underline{N}-p-1)}$ associated with $A_{(p+1,\underline{N}-p-1)} : A(N \times \{0,1\}, K)_{p+1} \to F(N_{\underline{N}-p-1}, Y)$, equation (2) implies that a linear map $B^{-1} \cdot A^b$ is a bijective linear map : $A(N \times \{0,1\}, K)_{p+1}/S \to Z$. For any $\lambda \in A(N \times \{0,1\}, K)_p \cap S$, $A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by injection of $B^{-1}A^b$. Hence $A_{(p+1,\underline{N}-p-1)}(S_r \lambda) = \underline{S_l} A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by using 2) in Lemma (7-C.5). This implies that $S_r \lambda \in S$. Therefore, S satisfies the condition 1 in Proposition (7-C.2).

Then Proposition (7-C.2) implies that a free motion with an affine map $(A(N \times \{0,1\}, K)_{p+1}/S, \tilde{S_r}), [e_{(0,0)}], [\bar{\eta}])$ is p-quasi-reachable. Where $[\bar{\eta}] = e_{(0,1)} - e_{(0,0)} + S$.

Here, equation (1) implies that there exists $x \in \text{im } A_{(p,\underline{N}-p-1)}$ such that $\underline{j}(x) = \underline{S_l} \cdot j(z)$ for any $z \in Z$. Moreover, by surjection of B, there exists $z' \in Z$ such that $B(z') = x$. Hence, $\underline{S_l} \cdot j(z) = \underline{j}(x) = \underline{j} \cdot B(z') = \pi \cdot j(z')$, which implies that im $(\underline{S_l} \cdot j) \subseteq$ im $(\pi \cdot j)$.
It follows that Z satisfies condition 4 in Proposition (7-C.4) and $((Z, F), 0)$ is $(\underline{N}-p-1)$-observable. Where F is given by the equation $Fz = (\pi \cdot j)^{-1} \underline{S_l} \cdot j(z)$ for any $z \in Z$. We can also show that $B^{-1} \cdot A^b$ is a free motion morphism $: A(N \times \{0,1\}, K)_{p+1}/S, \tilde{S_r}) \to (Z, F)$, and that an Almost Linear System $\sigma_1 = (A(N \times \{0,1\}, K)_{p+1}/S, \tilde{S_r}), [e(0,0)], [\bar{\eta}], 0 \cdot B^{-1} \cdot A^b, a(1))$ is isomorphic to an Almost Linear System $\sigma_2 = ((Z, F), B^{-1} \cdot A^b([e(0,0)]), B^{-1} \cdot A^b([\bar{\eta}]), 0, a(1))$. It follows that σ_1 and σ_2 are the natural partial realizations of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Hence, there exist natural partial realizations of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$.

(7-C.10) Lemma
Two canonical Almost Linear Systems are isomorphic if and only if their behaviors are the same.

[proof] This can be obtained from Theorem (7.8) and Corollary (7.9).

(7-C.11) proof of Theorem (7.25)
Let $A_{(\ ,\ )}$ be the partial linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. In order to prove necessity, we assume existence of the natural partial realization of $\underline{a}$.
Let Theorem (7.24) hold for integers p and p' that are different. Namely,
(1) im $A_{(p,\underline{N}-p-1)} =$ im $A_{(p+1,\underline{N}-p-1)}$
(2) ker $A_{(p,\underline{N}-p)} =$ ker $A_{(p,\underline{N}-p-1)}$
(3) im $A_{(p',\underline{N}-p'-1)} =$ im $A_{(p'+1,\underline{N}-p'-1)}$
(4) ker $A_{(p',\underline{N}-p')} =$ ker $A_{(p',\underline{N}-p'-1)}$
Then Propositions (7-C.6) and (7-C.7) imply that the dimension of $Z =$ im $A_{(p,\underline{N}-p-1)}$ is equal to one of $Z' =$ im $A_{(p',\underline{N}-p'-1)}$. Let σ and σ' be the natural partial realizations of $\underline{a}$ whose state spaces is Z and Z' respectively and which can be obtained by the same procedure as in (7-C.9). Then σ_1 is clearly isomorphic to σ' and the behavior of σ is equal to one of σ' by Lemma (7-C.10). This implies that the behavior of the natural partial realization is always the same regardless of different integers p and p'. Therefore,

the natural partial realization of $\underline{a}$ is unique modulo isomorphism by Lemma (7-C.10).
Next, we show sufficiency by the contrapositive. We assume that there does not exist a natural partial realization of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then minimum dimensional partial realization σ of $\underline{a}$ is p-quasi-reachable and q-observable for $p+q \geq \underline{N}$. It cannot be quasi-reachable within $p-1$ and not be observable within $q-1$. Then there exists a state x in σ such that x can be firstly reachable by an input ω with length p. The remaining data of $F(\Omega_{\underline{N}-p-1}, Y)$ can't determine a new state Fx because of $\underline{N}-p-1 < q$. Therefore, we can't determine the transition matrix F uniquely by q-observability. This implies that the minimum dimensional realization of $\underline{a}$ is not unique.

(7-C.12) proof of Theorem (7.26)
Let's consider the natural partial realization $\sigma_2 = ((Z, F), B^{-1} \cdot A^b([e(0,0)]), B^{-1} \cdot A^b([\bar{\eta}]), \underline{0}, a(1))$ of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ given in (7-C.9). Then we can obtain the quasi-reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, \bar{g}_s, h_s, a(1))$ from σ_2 in the same manner as the theorem for a Realization Procedure (7.21) (also see (7-B.19)).

7.7.D Real Time Partial Realization of Almost Linear Systems

(7-D.1) proof of Lemma (7.29)
Note that jth component of column vectors $\underline{S_l}^i(\chi^0) \in K^{L-1}$ is $(\underline{S_l}^i(\chi^0)(j-1) = a(0^{i+j}) - a(0^{i+j-1})$. Then $\{\underline{a}(0^k|\omega_1);$ any $k \in N\}$ can be uniquely determined by linear dependence. Since $\underline{a}$ is time-invariant, $\underline{a}(0^{k-1}|1|\omega_1) - \underline{a}(0^k|\omega_1) + \underline{a}(0^k) - \underline{a}(0^{k-1}) = \underline{a}(0^{k-1}|1) - \underline{a}(0^{k-1}) = (\underline{S_l}^i(\chi^0 + \bar{\chi}))(j-1)$ can be obtained for $i+j=k$. Therefore, if we add a further input $\omega_2 = 0^{L_2+L-1}|1$, step 2) can be inspected. Since jth component of row vectors $\underline{S_l}^i(\chi^0+\bar{\chi}) \in K^{L-1}$ is $(\underline{S_l}^i(\chi^0+\bar{\chi}))(j-1) = \underline{a}(0^{i+j}|1|\omega_1) - \underline{a}(0^{i+j}|\omega_1) + \underline{a}(0^{i+j}) - \underline{a}(0^{i+j-1})$, $\{\underline{a}(0^k|\omega_2|\omega_1);$ any $k \in N\}$ can be uniquely determined by linear dependence. Then a partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ is obtained. Since the physical object is less than L dimensional, the partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ contains the whole input/output data by (7-C.9).

8 So-Called Linear Systems

Almost Linear Systems having been introduced in Chapter 7, the facts about the systems were cleared, and it was also shown that the systems contain So-called Linear Systems as a sub-class.
Where So-called Linear Systems are linear systems with a non-zero initial state.

In this chapter, based on the these results of the systems, we discuss the finite dimensionality, the partial realization and real-time partial realization of So-called Linear Systems. It is well known that a usual method to obtain So-called Linear Systems is solved through two problems.
One is the realization problem to obtain linear systems with a zero initial state and the other is the state estimation problem for the systems with a non-zero initial state. Based upon the prejudice that So-called Linear Systems are completely the same as linear systems, So-called Linear Systems have not been treated seperately.

This chapter presents a new method to obtain them directly from input/output data on real-time. This can be done because of the following two reasons. One is that Almost Linear Systems are obtained by real-time data (equivalently, by a single-experiment) and it also will be shown that So-called Linear Systems are in sub-class of Almost Linear Systems. The second reason is that it will be made clear what the behaviour of So-called Linear Systems is. Thus, So-called Linear Systems are obtained by real time data.

It is generally known that partial realization problems are very important for simulation and control, and it is known that a solution of the problems for non-linear systems requires multi-experiments. The experiment must be only done with many objects of consideration. However, this situation is not practical. Therefore, a partial realization problem by single-experiment (equivalently, a real-time partial realization problem) is very useful and practical. Hence, this new method is more useful and practical than usual.

The partial realization problem of the systems can be stated as follows:
[Let I/O be the special sub set of any input/output map which satisfies causality, time-invariance and affinity and CD be the category of canonical (controllable & observable) So-called Linear Systems, then obtain the following real-time partial realization theorem.]

Real-time partial Realization Theorem: $<$ For data of any given input/output map $a \in I/O$ obtained on real-time (equivalently, by single experiment), there exists a minimal dimensional dynamical system which has the same real-time partial behavior a. For any minimal dimensional systems σ_1

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and σ_2 which have the same partial behavior a, σ_1 is isomorphic to σ_2 in the sense of category CD. Moreover, there exists an algorithm to obtain a So-called Linear System which partially realizes (describes) a.>

8.1 Input Output Relations for So-Called Linear Systems

First we will clarify when I/O is equal to the set of behaviors for So-called Linear Systems. Then we will solve the real time partial realization problem.

Let the output value's set Y be any linear space over the field K. Especially, let $Y = K^p$.

(8.1) Definition
Let an input response map $a \in F(U^*, Y)$ satisfy the following time-invariance and affinity condition, then a is said to be a time-invariant, affine input response map.
1) Time-invariant condition:
$a(\omega_1|\omega) - a(\omega_1) = a(\bar{\omega}_1|\omega) - a(\bar{\omega}_1)$
for any ω, and $\omega_1, \bar{\omega}_1$ such that $|\omega_1| = |\bar{\omega}_1|$.

2) Affinity condition:
$a : \Omega \to Y$ is an affine map, i.e.,
$a(\omega + \bar{\omega}) + a(0^{|\omega|}) = a(\omega) + a(\bar{\omega})$
$a(\lambda\omega) = \lambda a(\omega) + (1 - \lambda)a(0^{|\omega|})$
for any $\omega, \bar{\omega} \in \Omega, |\omega| = |\bar{\omega}|$ and $\lambda \in K$.
Where 0^t denotes $0^t(s) = 0$ for any $s(0 \leq s \leq t)$.

(8.2) Definition
For a time-invariant, affine input response map $a \in F(U^*, Y)$, the following function GI_a is said to be a modified impulse response of a.
$GI_a : \{0, 1\} \to F(N, Y); u \mapsto GI_a(u)[; t \mapsto a(u^{t+1}) - a(u^t)]$

(8.3) Theorem
For any time-invariant, affine input response map $a \in F(U^*, Y)$, there uniquely exists the modified impulse response $GI_a : \{0, 1\} \to F(N, Y)$.
This correspondence is bijective.
Where the inverse correspondence is given by the following equation:
$a(\omega) = a(\mathit{1}) + \sum_{j=1}^{|\omega|} \omega(j)(GI_a(1)(n-j)) + (1-\omega(j))(GIa(0)(n-j)), \omega \in U^*.$

[proof] This is the same as Theorem (7.3) in Chapter 7.

This equation between a and GI_a is as same as the equation between input/output map and impulse map in linear systems. The equation in linear systems is represented by the convolution.

8.2 So-Called Linear Systems and Almost Linear Systems

To be concluded in this chapter, we will briefly introduce Almost Linear Systems which realize (faithfully describe) any time-invariant, affine input response map $a \in F(U^*, Y)$. Where the systems have been discussed in Chapter 7 in details.

(8.4) Definition
A system given by the following equations is written as a collection $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$ and it is said to be an Almost Linear System.

$$\begin{cases} x(t+1) &= Fx(t) + g^0 + \bar{g}\omega(t+1) \\ x(0) &= 0 \\ \gamma(t) &= h^0 + hx(t) \end{cases}$$

Where X is a linear space over the field K, $F \in L(X)$ and $\omega(t) \in U$ for any $t \in N$. And $g^0, g \in X$, h is a linear operator : $X \to Y$ and $h^0 \in Y$.
For an Almost Linear System $\sigma = ((X, F), g^0, \bar{g}, h, h^0)$,
a function $a_\sigma : U^* \to Y; \omega \mapsto h^0 + h(\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j)))$ is said to be a behavior of the Almost Linear System σ.
σ can be represented by the following diagram Fig. 8.1.
If $a_\sigma = a$ holds, then a system σ is called a realization of a.
A system σ is said to be quasi-reachable if X is equal to $\ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j)) : \omega \in U^*\} \gg$.
A system σ is called reachable if X is equal to $\{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + \bar{g}\omega(j)) : \omega \in U^*\}$.
A system σ is called observable if $hF^j x = 0$ for any $j \in N$ implies $x = 0$.
A system σ is called canonical if σ is quasi-reachable and observable.
A system σ is said to be intrinsically canonical if σ is reachable and observable.

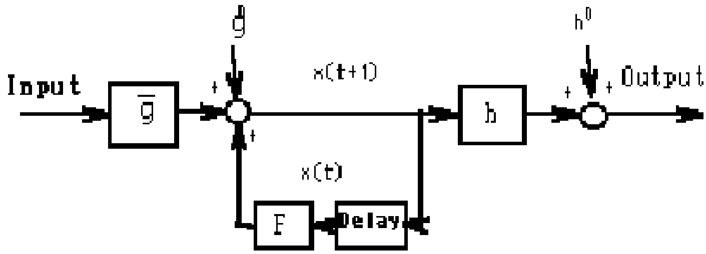

Fig. 8.1. A block diagram of an Almost Linear System $\sigma = ((K^n, F), , g^0, \bar{g}, h, h^0)$

Remark: Notice that a canonical Almost Linear System σ is a system which has the most reduced state space X among Almost Linear Systems that have the behavior a_σ.

(8.5) Definition
A system given by the following system equation is said to be a So-called Linear System $\tilde{\sigma} = ((X, F), x^0, g, h)$. This system is a linear system with a non-zero initial state.

$$\begin{cases} x(t+1) & = & Fx(t) + g\omega(t+1) \\ x(0) & = & x^0 \\ \gamma(t) & = & hx(t) \end{cases}$$

Where $F \in L(X)$, $\omega(t+1) \in U$, $g \in X$. And h is a linear operator : $X \to Y$.

(8.6) Proposition
An Almost Linear System $\sigma = ((K^n, F), g^0, \bar{g}, h, h^0)$ is intrinsically canonical if and only if the following two conditions hold.
rank $[\bar{g}, F\bar{g}, F^2\bar{g}, \cdots, F^{n-1}\bar{g}] = n$
rank $[h^T, (hF)^T, \cdots, (hF^{n-1})^T] = n$.

[proof] This can be easily obtained from the Definition (8.4).

(8.7) Proposition
For any So-called Linear System $\tilde{\sigma} = ((X, F), x^0, g, h)$, there exists an Almost Linear System $\sigma = ((X, F), g^0, g, h, h^0)$ with the same input/output relation which satisfies $g^0 = Fx^0 - x^0$ and $h^0 = hx^0$.

[proof] This can be proved by direct calculations.

(8.8) Lemma
Let $\tilde{\sigma} = ((X, F), x^0, g, h)$ be a canonical (controllable and observable) So-called Linear System, then the Almost Linear System σ obtained by Proposition (8.7) is intrinsically canonical.
Conversely, let $\sigma = ((X, F), g^0, g, h, h^0)$ be an intrinsically canonical Almost Linear System, then So-called Linear System $\tilde{\sigma}$ obtained by Proposition (8.7) is canonical.

[proof] By definition of intrinsic canonicality and canonicality, this is easily obtained.

(8.9) Proposition
Let $\sigma = ((X, F), g^0, g, h, h^0)$ be an intrinsically canonical Almost Linear System. A canonical So-called Linear System $\tilde{\sigma} = ((X, F), x^0, g, h)$ is given by σ if and only if there exists $x^0 \in X$ such that $g^0 = Fx^0 - x^0$.

[proof] By Proposition (8.7) and Lemma(8.8), we can obtain this proposition.

(8.10) Example (An Almost Linear System which is not a So-called Linear System)
Let's consider an intrinsically canonical Almost Linear System $\sigma =$ $((R^3, F), g^0, g, h, h^0)$. Where $g^0 = g = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $h = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$,
$$F = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$
Then there does not exist $x^0 \in X$ such that $g^0 = Fx^0 - x^0$. Hence, by Proposition (8.9), this system is an Almost Linear System which is not a So-called Linear System.

(8.11) Definition

Let $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ be a canonical Almost Linear System which is given by the followings. Then σ_s is said to be a reachable standard system. $g_s^0 = [\times \times \cdots \times \cdots \times]^T$, $g_s = [100 \cdots 00]^T$, $h_s = [\times \times \cdots \times \cdots \times]$.

$$F_s = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & \alpha_1 \\ 1 & 0 & & & \vdots & \alpha_2 \\ 0 & 1 & \ddots & & \vdots & \alpha_3 \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \alpha_n \end{bmatrix}.$$

Note that F_s in the reachable standard system σ_s is equal to the transition matrix of a canonical linear system, and the system σ_s is reachable.

(8.12) Definition

Let $\sigma_s = ((K^n, F_s), x_s^0, g_s, h_s)$ be a So-called Linear System which is given by the following. Then σ_s is said to be a reachable standard system. $x_s^0 = [\times \times \cdots \times \cdots \times]^T$, $g_s = [100 \cdots 00]^T$, $h_s = [\times \times \cdots \times \times]$. And F_s is the same as F_s in Definition (8.11).

(8.13) Theorem

For any intrinsically canonical Almost Linear System $\sigma = ((K^n, F), g^0, g, h, h^0)$, there exists a unique reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ with the same behaviour which is isomorphic to it.

[proof] See (8-A.1) in Appendix 8.6.

(8.14) Theorem

Let $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ be the reachable standard system. Then the canonical So-called Linear System $\tilde{\sigma}$ can be obtained by σ_s if and only if there exist $x_i \in K (1 \le i \le n)$ such that $g_s^0 = [-x_1 + \alpha_1 x_n,\ x_1 - x_2 + \alpha_2 x_n,\ x_2 - x_3 + \alpha_3 x_n,\ \cdots,\ x_{n-1} - x_n + \alpha_n \cdot x_n]^T$.

Where $\alpha_i \in K$, see Definition (8.11).

[proof] See (8-A.2) in Appendix 8.6.

In Chapter 7, we introduced the following Input/Output Matrix for Almost Linear Systems, which is said to be the Input/Output Matrix.
Where $u \in \{0,1\}$ and $s, t \in N$.

$$(I/O)_a = \begin{array}{c} \\ \\ \\ \\ t \end{array} \left(\begin{array}{ccc} & & (s,u) \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(u^{s+t+1}) - a(u^{s+t}) \\ & & \end{array} \right)$$

Remark: Note that the column vectors in $(I/O)_a$ denote $S_l^s \chi(u) \in F(N,Y)$, $(S_l^s \chi(u))(t) = a(u^{s+t+1}) - a(u^{s+t})$.

Here we will introduce a new input/output matrix to be suitable for So-called Linear Systems.

(8.15) Definition
For any time-invariant, affine input response map $a \in F(U^*, Y)$, the following infinite matrix H_a^S is said to be a Hankel matrix. Where $u \in \{0,1\}$, $s, t \in N$.

$$H_a^S = \begin{array}{c} \\ \\ \\ \\ t \end{array} \left(\begin{array}{ccc} & & s \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(u^{s+t}|1) - a(u^{s+t}|0) \\ & & \end{array} \right)$$

Remark: Note that the column vectors in H_a^S denote $GI_a(1)(s) - GI_a(0)(s)$. Simultaneously, it also denotes $S_l^s \bar{\chi} \in F(N,Y)$, where $S_l^s \bar{\chi}(t) = a(\omega|1) - a(\omega|0)$, $t = |\omega|$.

(8.16) Theorem
For a time-invariant, affine input response map $a \in F(U^*, Y)$, the following conditions are equivalent:
(1) a is a behavior of an intrinsically canonical n dimensional Almost Linear System.
(2) There exist n linearly independent vectors and no more than n vectors in $\{S_l^i(\chi^0 + \bar{\chi} \cdot u) : i \in N, u \in \{0,1\}\}$.
Especially, there exist n independent vectors in $\{S_l^i \bar{\chi} : i \in N\}$. For $\bar{\chi}$ and χ^0, see Example (7.5) in Chapter 7 .

(3) The rank of the Input/Output Matrix $(I/O)_a$ of a is n. Especially, the rank of Hankel matrix H_a^S of a is n.

[proof] See (8-A.3) in Appendix 8.6.

The following theorem will clearly denote a relation between time-invariant, affine input response maps and the behaviors of So-called Linear Systems.

(8.17) Theorem
For a time-invariant, affine input response map $a \in F(U^*, Y)$, the following conditions are equivalent:
(1) a is a behavior of a canonical n dimensional So-called Linear System.
(2) rank $(I/O)_a$ = rank $H_a^S = n$ holds. And there exist coefficients $\{x_i \in K : 1 \leq i \leq n\}$ such that the following equation holds:
$[a(0) - a(1), a(0^2) - a(0), , a(0^{n-1}) - a(0^{n-2})]^T = \sum_{i=1}^{n} x_i(S_l^i \underline{\bar{\chi}} - S_l^{i-1} \underline{\bar{\chi}})$.
Where $S_l^i \underline{\bar{\chi}}$ denotes the vector
$[a(0^i|1) - a(0^{i+1}), a(0^{i+1}|1) - a(0^{i+2}), \cdots, a(0^{i+n-1}|1) - a(0^{i+n})]^T \in K^n$.

[proof] See (8-A.4) in Appendix 8.6.

(8.18) Definition
If a time-invariant, affine input response map $a \in F(U^*, Y)$ satisfies the condition of Theorem (8.17), then a is said to be a So-called linear input response map.

(8.19) Theorem for a Realization Procedure
Let a time-invariant, affine input response map $a \in F(U^*, Y)$ satisfy the conditions of Theorem (8.17), then the reachable standard system
$\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ which realizes a can be obtained by the following procedure:
1) Select the n linearly independent vectors $\{S_l^i \chi : 0 \leq i \leq n-1\}$.
Where $n :=$ rank H_a^R.
2) Let the state space be K^n. Let $g_s^0 \in K^n$ be
$g_s^0 = [g_1, g_2, \cdots g_n]^T$ and $g_s = e_1 \in K^n$.
Where $\chi^0 = \sum_{i=1}^{n} g_i S_l^i$, $g_i \in K.(1 \leq i \leq n)$.
3) Let the output map h_s be the following:
$h_s = [a(1) - a(0), a(0|1) - a(0^2), \cdots, a(0^{n-1}|1) - a(0^n)]$.
4) Let F_s be the matrix in Definition (8.11).

Where $S_l^n \bar{\chi} = \sum_{i=1}^n \alpha_i S_l^{i-1} \bar{\chi}$, $\alpha_i \in K$.

[proof] See (8-A.5) in Appendix 8.6.

Checking condition (2) in Theorem (8.17), we will clarify whether the reachable standard system σ_s obtained by Theorem (8.19) is the canonical So-called Linear System. We will obtain the following lemma.

(8.20) Lemma
For a So-called linear input response map $a \in F(U^*, Y)$, the reachable standard system obtained by Theorem (8.19) is a So-called Linear System.

[proof] This is obvious from Theorem (8.19).

8.3 Partial Realization Theory of So-Called Linear Systems

Based on the foregoing results of So-called Linear Systems, we will consider a partial realization problem by multi-experiment for the systems.
Let $\underline{a}$ be an $\underline{N}$ sized So-called linear input response map ($\in F(\Omega_{\underline{N}}, Y)$), where $\underline{N} \in N$ and $\Omega_{\underline{N}} := \{\omega \in \Omega; |\omega| \leq \underline{N}\}$. Then $\underline{a}$ is said to be a partial So-called linear input response map. Note that the partial So-called linear input response map $\underline{a}$ is a special partial time-invariant, affine input response map.

For a partial So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, a finite dimensional So-called Linear System $\sigma = ((X, F), x^0, g, h)$ is said to be a partial realization of $\underline{a}$ if $a_\sigma : \Omega \to Y; \omega \mapsto h^0 + h(F^{|\omega|} + \sum_{j=1}^{|\omega|}(F^{|\omega|-j} g\omega(j)) = \underline{a}(\omega)$ holds for any $\omega \in \Omega$, $|\omega| \leq \underline{N}$.

A partial realization problem of So-called Linear Systems can be stated as follows:
$<$ For any given partial So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, find a partial realization σ of $\underline{a}$ such that the dimensions of state space X of σ is minimum, where the σ is said to be a minimal partial realization of $\underline{a}$. Moreover, show when the minimal realizations are isomorphic.$>$

In Section 1 of Chapter 7, we have obtained the representation theorem for the time-invariant, affine input response maps. The theorem says that any time-invariant, affine input response map can be characterized by the modified impulse response. Note that the modified impulse response

$GI : \{0,1\} \to F(N,Y)$ can be represented by $(GI_a(u)(t)) = a(u^{t+1}) - a(u^t)$ for $u \in \{0,1\}$, $t \in N$ and the time-invariant, affine input response map $a \in F(U^*,Y)$ (equivalently, $a \in F(\Omega,Y)$). Moreover, for any given partial time-invariant, affine input response $\underline{a} \in F(\Omega_{\underline{N}},Y)$, this correspondence can uniquely produce a partial modified impulse response $GI_{\underline{a}} : \{0,1\} \to F(N_{\underline{N}},Y)$.
Where $N_{\underline{N}} := \{1,2,\cdots,\underline{N};$ for some $\underline{N} \in N\}$.
Hence, by Theorem (8.3), for any partial So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}},Y)$, there will exist a unique partial modified impulse response $GI_{\underline{a}} : \{0,1\} \to F(N_{\underline{N}},Y)$.

(8.21) Proposition
For any given So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}},Y)$, there always exists a minimal partial realization of it.

[proof] This proof can be obtained the same as Proposition (7.22) in Chapter 7.

Minimal partial realizations are in general not unique modulo isomorphism. Therefore, we will introduce a natural partial realization, and we will show that natural partial realizations exist if and only if they are isomorphic.

(8.22) Definition
Let $\sigma = ((X,F),x^0,g,h)$ be a So-called Linear System and $p \in N$ be some integer. If $X = \{\sum_{j=1}^{|\omega|}(F^{|\omega|-j}g\omega(j); \omega \in \Omega_p\}$ holds, then σ is said to be p-reachable.
Let q be some integer. If $hF^{q'}x = 0$ for any $q' \leq q$ implies $x = 0$, then σ is said to be q-observable.
Let $\underline{a} \in F(\Omega_{\underline{N}},Y)$ be a given partial So-called linear input response map. If there exists σ which is p-reachable and q-observable such that $p + q < \underline{N}$, for some $p,q \in N$ then σ is said to be a natural partial realization of $\underline{a}$.
For a partial time-invariant, affine input response map $\underline{a} \in F(\Omega_{\underline{N}},Y)$, the following matrix $(I/O)_{\underline{a}\ (p,\underline{N}-p)}$ is said to be a finite-sized Input/Output Matrix of $\underline{a}$.

$$\underset{t}{(I/O)_{\underline{a}\ (p,\underline{N}-p)}} = \begin{pmatrix} & & (s,u) \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(u^{s+t+1}) - a(u^{s+t}) \\ & & \end{pmatrix}$$

Where $0 \leq s \leq p$, $0 \leq t \leq \underline{N} - p$ and $u \in \{0, 1\}$.

(8.23) Definition

For any time-invariant, affine input response map $a \in F(U^*, Y)$ (equivalently, $a \in F(\Omega, Y)$), the following bounded matrix is said to be a bounded Hankel matrix $H^S_{a\ (p,M-p)}$.

The matrix can be obtained by multi-experiment on $a \in F(U^*, Y)$. Namely, this matrix can be derived by $GI_a(u) \in F(\mathbf{M}, Y)$, where $\mathbf{M} = \{0, 1, , M\}$ for $M \in N$. Where $u = 0$ or $u = 1$. And $0 \leq s \leq p$, $0 \leq t \leq M - p$.

$GI_a(u)$ is said to be a partial modified impulse response of $a \in F(U^*, Y)$.

$$\underset{t}{H^S_{a(p,M-p)}} = \begin{pmatrix} & & s \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ \cdots & \cdots & a(u^{s+t}|1) - a(u^{s+t}|0) \\ & & \end{pmatrix}$$

Note that column vectors of $H^S_{a(p,M-p)}$ may be represented by $S^s_l \underline{\bar{\chi}}$, or equivalently, $GI_a(1)(s) - GI_a(0)(s)$.

(8.24) Theorem

Let $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ be a partial time-invariant, affine input response map. Then there exists a natural partial realization of $\underline{a}$ which is intrinsically canonical if and only if the following conditions hold.

rank $(I/O)_{a(p,\underline{N}-p)}$ = rank $(I/O)_{a(p,\underline{N}-p-1)}$ = rank $(I/O)_{a(p+1,\underline{N}-p-1)}$

= rank $H^S_{a(p,\underline{N}-p)}$ = rank $H^S_{a(p,\underline{N}-p-1)}$ = rank $H^S_{a(p+1,\underline{N}-p-1)}$ for some $p \in N$.

[proof] See (8-B.6) in Appendix 8.6.

(8.25) Theorem
For a partial So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$, the following conditions are equivalent:
(1) $\underline{a}$ is a behavior of a natural partial realization which is a So-called Linear System.
(2) rank $(I/O)_{a(p,\underline{N}-p)} =$ rank $(I/O)_{a(p,\underline{N}-p-1)} =$ rank $(I/O)_{a(p+1,\underline{N}-p-1)}$ $=$ rank $H^S_{a(p,\underline{N}-p)} =$ rank $H^S_{a(p,\underline{N}-p-1)} =$ rank $H^S_{a(p+1,\underline{N}-p-1)}$ for some $p \in N$ holds. And there exist coefficients $\{x_i \in K : 1 \leq i \leq n\}$ such that the following equation holds:
$[\underline{a}(0) - \underline{a}(1), \underline{a}(0^2) - \underline{a}(0), \cdots, \underline{a}(0^p) - \underline{a}(0^{p-1})]^T = \sum_{i=1}^{n} x_i(\underline{S_l}^i \underline{\bar{\chi}} - \underline{S_l}^{i-1} \underline{\bar{\chi}})$.
Where $\underline{S_l}^i \underline{\bar{\chi}}$ denotes the vector
$[\underline{a}(0^i|1) - \underline{a}(0^{i+1}), \underline{a}(0^{i+1}|1) - \underline{a}(0^{i+2}), \cdots, a(0^{i+p-1}|1) - a(0^{i+p})]^T$.

[proof] See (8-B.7) in Appendix 8.6.

(8.26) Theorem
There exists a natural partial realization of a given partial So-called linear input response map $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ if and only if the minimal partial realizations of $\underline{a}$ are unique modulo isomorphism.

[proof] See (8-B.8) in Appendix 8.6.

(8.27) Theorem
Let a partial time-invariant, affine input response $\underline{a} \in F(\Omega_{\underline{N}}, Y)$ satisfy the condition of Theorem (8.25), then the reachable standard system $\sigma = ((X, F_s), x_s^0, g_s, h_s)$ which realizes $\underline{a}$ can be obtained by the following algorithm.
Set $n :=$ rank $H_{a(p,\underline{N}-p)}$, where $H_{a(p,\underline{N}-p)}$ is the finite Hankel matrix of $\underline{a} \in F(\Omega_{\underline{N}}, Y)$.
1) Select the linearly independent vectors $\{\underline{S_l}^i \underline{\bar{\chi}}; 1 \leq i \leq n)\}$ from the column vectors of $H_{a(p,\underline{N}-p)}$.
2) Let the state space be K^n. Let the map $g_s = e_1$.
3) Let the output map $h_s = [\underline{a}(1) - \underline{a}(0), \underline{a}(0|1) - \underline{a}(0^2), \cdots, \underline{a}(0^{n-1}|1) - \underline{a}(0^n)]$
4) Let F_s be the matrix in Definition (8.11).
Where $\underline{S_l}^n \underline{\bar{\chi}} = \sum_{i=1}^{n} \alpha_i \underline{S_l}^{i-1} \underline{\bar{\chi}}$, $\alpha_i \in K$.
And $\underline{S_l} : F(N_s, Y) \to F(N_{s-1}, Y); \underline{a} \mapsto \underline{S_l}\underline{a}[; t \mapsto \underline{a}(t+1)]$ for some $s \in N$.

[proof] See (8-B.9) in Appendix 8.6.

8.4 Real-Time Partial Realization of So-Called Linear Systems

In general, it is known that non-linear systems can be only determined by multi-experiments. In fact, in Chapter 3, a condition for a general unknown black-box to be determined with single-experiment was given. This condition may be very hard for us to find in practice. However, we could look for special single-experiments to pretend multi-experiments for any Almost Linear System. So-called Linear Systems being the special class of Almost Linear Systems, we will be able to solve the real-time partial realization problem for So-called Linear Systems.

In this section, on the results of the partial realization theory in section 8.3, we will discuss a single-experiment for So-called Linear Systems.

(8.28) Real-time partial realization problem
Let a physical object (equivalently, $a \in F(\Omega, Y)$) be a finite dimensional So-called Linear System. Then for given finite data $\{a(\omega); \omega$ is an input with a finite length $\}$, find the So-called Linear System $\sigma = ((X, F), x^0, g, h)$ and an input $\bar{\omega}$ such that $a_\sigma(\omega) = a(\omega)$ for any $\omega \in \Omega$.

(8.29) Definition
For a finite dimensional So-called Linear System, if there exists a solution of the real time partial realization problem, then an input $\bar{\omega}$ of the solution is said to be a (real time partial) realization signal.

(8.30) Lemma
Let a So-called linear input response map $a \in F(\Omega, Y)$ have the behavior of a So-called Linear System whose state space is less than L-dimensional. Then there exists an input of finite length $\bar{\omega} \in \Omega$ such that the following algorithm provides a finite Input/Output Matrix.
Where $p := max\{L_1, L_2\}$.
1) Find an integer L_1 such that row vectors $\{\underline{S_l}^i \underline{\chi}^0 \in K^L; 0 \le i \le L_1 - 1\}$ are linearly independent and $\{\underline{S_l}^i \underline{\chi}^0 \in K^L; 0 \le i \le L_1\}$ are linearly dependent. Namely, feed an input $\omega_1 := 0^{L_1+L+1}$ into the plant.
Where $\underline{S_l}^i \underline{\chi}^0 = [a(0^{i+1}) - a(0^i), a(0^{i+2}) - a(0^{i+1}), \cdots, a(0^{L+i+1}) - a(0^{L+i})]^T$.
2) Find an integer L_2 such that row vectors $\{\underline{S_l}^i \underline{\chi} \in K^L; 0 \le i \le L_2 - 1\}$ are linearly independent and $\{\underline{S_l}^i \underline{\chi} \in K^L; 0 \le i \le L_2\}$ are linearly dependent. Namely, feed a further input $\omega_2 := 0^{L_2+L-1}|1$ into the plant. Let $\bar{\omega} = \omega_2|\omega_1$.

Where $\underline{S_l}^i\underline{\chi} = [a(0^i|1) - a(0^{i+1}), a(0^{i+1}|1) - a(0^{i+2}), \cdots, a(0^{i+L}|1) - a(0^{i+L+1})]^T \in K^L$.
And $a(0^i|1)$ is given by $a(0^j|1) = a(0^{i+1}|1|0^t) - a(0^{t+1}) + a(0^t)$ for any $i, t \in N$.
Making the row vectors of a matrix from the row vectors $\{\underline{\chi}^0, \underline{S_l}^i\underline{\chi} \in K^{L-1};\ 0 \leq i \leq L_2\}$ obtained by the above iterations, we will obtain a finite Input/Output Matrix $H^S_{a(L-1,p)}$.

[proof] See (8-C.1) in Appendix 8.6.

(8.31) Theorem
Let a So-called linear input response map $a \in F(\Omega, Y)$ have the behavior of a So-called Linear System whose state space is less than L dimensional. Then there exists a realization signal such that the reachable standard system $\sigma = ((X, F_s), g_s^0, g_s, h_s)$ which realizes a can be obtained by the following algorithm:
1) Find a finite Input/Output Matrix $H^S_{a(L-1,p)}$ upon the algorithm given in Lemma (8.30).
2) Apply the algorithm given in Theorem (8.27) to the above finite Input/Output Matrix $(I/O)_{a(L-1,p)}$.

[proof] This can be obtained by Lemma (8.30).

8.5 Historical Notes and Concluding Remarks

Based on the results of Almost Linear Systems which contain So-called Linear Systems as a subclass, we understood a clear relation between the two systems and solved the real-time partial realization problem for So-called Linear Systems. We have presented a new method of obtaining them on real-time (equivalently, by single experiment). This method is more practical and useful than the previous one. Note that a usual realization problem must be solved by two problems. One is the realization problem of linear systems, and the other is the state estimation problem of linear systems with a non-zero initial state. The usual method is only used in a very convenient situation, but not in an actual situation for solving a partial realization problem.

This solution can be derived from the fact that So-called Linear Systems are non-linear systems, not linear systems and the systems are in the subclass of Almost Linear Systems. Moreover, the special class of Almost Linear Systems was made clear.

8.6 Appendix

8.6.A So-Called Linear Systems and Almost Linear Systems

In this Appendix, we will give proofs for theorems, propositions and corollaries stated in section 8.2.

(8-A.1) Proof of Theorem (8.13)
Let's consider an Almost Linear System morphism:
$T : \sigma = ((K^n, F), g^0, g, h, h^0) \to \sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ in proof of Theorem (7.16). Then T is bijective and σ_s is the only quasi-reachable standard system. Since bijection of T preserves independence and dependence in σ to ones in σ_s. By definition of reachablity and reachable standard system in Definitions (8.4) and (8.11), σ_s is a unique reachable standard system.

(8-A.2) Proof of Theorem (8.14)
According to Theorem (8.13), without loss of generality, any intrinsically canonical Almost Linear System can be considered as the reachable standard system. In the reachable standard system, we will prove this theorem. Let $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ be the reachable standard system. Then by Proposition (8.9), σ_s is a So-called Linear System if and only if there exists an initial state $x^0 \in K^n$ such that $[F_s - I]x^0 = g_s^0$. Hence, this condition is equal to rank $[F_s - I] =$ rank $[F_s - I, g_s^0]$. This rank condition means $g_s^0 \in$ im $[F_s - I]$. Therefore, it follows that there exist coefficients $\{x_i \in K : 1 \leq i \leq n\}$ such that $g_s^0 = [-x_1 + \alpha_1 x_n, x_1 - x_2 + \alpha_2 x_n, x2 - x3 + \alpha_3 x_n, \cdots, x_{n-1} - x_n + \alpha_n x_n]^T$.

(8-A.3) Proof of Theorem (8.16)
Let's consider the linear input/output map $A : (A(N \times \{0, 1\}, K), S_r) \to (F(N, Y), S_l)$ corresponding to $a \in F(\Omega, Y)$ discussed in Corollary (7-A.15) in Chapter 7. im $A = \{S_l^i(\chi^0 + \bar{\chi} \cdot u); u \in K, i \in N\}$ holds. Then an equality of (1) and (2) can be easily obtained by Theorem (7.20).
1) $\Longrightarrow$ 3). Since im A is n dimensional, definition of intrinsically canonicality and Proposition (8.6) imply that $\{S_l^i \bar{\chi} \cdot u); u \in K, i \in N\}$ is specially n dimensional. Hence, the rank of the Hankel matrix corresponding to it is n.
3) $\Longrightarrow$ 2). For $i, j \in N$ such that $1 \leq i, j \leq n$, let $[a(u^{i+j}|1) - a(u^{i+j}|0)]$ be $n \times n$ regular matrix. Since vectors $\{[a(u^{j+1}|1) - a(u^{j+1}|0)a(u^{j+2}|1) - a(u^{j+2}|0) \cdots a(u^{j+n}|1) - a(u^{j+n}|0)]^T : 1 \leq j \leq n\}$ are linearly independent, $\bar{\chi}$, $S_l\bar{\chi}$, $S_l^2\bar{\chi}$, $S_l^{n-1}\bar{\chi}$ are linearly independent. For any $s \in N$, let

$\lambda = [\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n]^T$ be determined such as the following. $[a(u^{i+j}|1) - a(u^{i+j}|0)]\lambda = [a(u^s|1) - a(u^s|0)a(u^{s+1}|1) - a(u^{s+1}|0) \cdots$
$a(u^{s+n-1}|1) - a(u^{s+n-1}|0)]^T$.
Then for any $j \in N$, the equation $S_l^s \bar{\chi}(j) = \sum_{i=1}^n \lambda_i S_l^{i-1} \bar{\chi}(j)$ holds. Hence, $S_l^s \bar{\chi} = \sum_{i=1}^n \lambda_i S_l^{i-1} \bar{\chi}$ can be obtained. Thus, the condition 2) is obtained.

(8-A.4) Proof of Theorem (8.17)
In Theorem (8.16), if we add a specialty of So-called Linear System to the theorem, then we will finish proving Theorem (8.17). Since the Almost Linear System $((\ll S_l^N(\chi(U)) \gg, S_l), \chi^0, \bar{\chi}, 0, a(1))$ in Theorem (7.6) realizes the time-invariant, affine input response map $a \in F(\Omega, Y)$, we may seek for the condition of Proposition (8.9) in this system. This equivalent condition is the condition (2) of this theorem.

(8-A.5) Proof of Theorem (8.19)
For $\{S_l^{i-1}\chi : 1 \le i \le n\}$ selected in the procedure (1), let a linear operator $T :\ll \{S_l^{i-1}\chi : 1 \le i \le n\} \gg \to K^n$ be $TS_l^{i-1}\chi = e_i$. For F_s determined by the procedure (4), we can obtain a relation $TS_l = F_s T$. By procedure (2), $T\chi^0 = g_s^0$ holds. By procedure (3), $0 = h_s T$ holds. Hence, a linear operator $T :\ll \{S_l^{i-1}\chi : 1 \le i \le n\} \gg \to K^n$ is an Almost Linear System morphism : $((\ll S_l^N(\chi(U)) \gg, S_l), \chi^0, \bar{\chi}, 0, a(1)) \to \sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, a(1))$. By Corollary (7.9), both behaviours are the same. Since the Almost Linear System $((\ll S_l^N(\chi(U)) \gg, S_l), \chi^0, \chi, 0, a(1))$ realizes a, σ_s realizes a. The fact that σ_s is the reachable standard system comes from its definition.

8.6.B Partial Realization

This appendix will be prepared for the proof of theorems and proposition stated in section 8.3. See Appendix 7.7 for details of notions and notations.

8.6.B.1 Free Motions with an Affine Map

We can consider a partial linear input/output map $A_{(p,\underline{N}-p)}$: $A(N \times \{0,1\}, K)_p \to F(N_{\underline{N}-p}, Y)$ for $\underline{a} \in F(\underline{N}, Y)$ the same as the linear input/output map $A : (A(N \times \{0,1\}, K), S_r) \to (F(N,Y), S_l)$ considered for any time-invariant, affine input response map $a \in F(\Omega, Y)$ in Corollary (7-A.15) of Chapter 7.
Where $A(N \times \{0,1\}, K)_p := \{\sum_{q,u} \lambda(q,u) e_{(q,u)} \in A(N \times \{0,1\}, K),\ q \le p$ for

$p \in N\}$, $F(N_q, Y) := \{\gamma \in F(N, Y)$; a function $\gamma : N_q \to Y\}$, and J_p is the canonical injection : $A(N \times \{0,1\}, K)_p \to A(N \times \{0,1\}, K)$. Let $H_q = P_q \cdot H$, where P_q is the canonical surjection : $F(N, Y) \to F(N_q, Y); \gamma \mapsto [; t \mapsto \gamma(t)]$, and $\underline{S_l}$ is given by setting $\underline{S_l} : F(N_q, Y) \to F(N_{q-1}, Y); \gamma \mapsto \underline{S_l}\gamma[; t \mapsto \gamma(t+1)]$.
Moreover, we can introduce a linear input/output map $A^S_{(p,\underline{N}-p)}$: $A(N \times \{0,1\}, K)^S_p \to F(N_{\underline{N}-p}, Y)$ for $A^S_{(p,\underline{N}-p)} = A_{(p,\underline{N}-p)} \cdot i_p$.
The $A^S_{(p,\underline{N}-p)}$ is said to be a partial reachable linear input/output map.
Where $A(N \times \{0,1\}, K)^S_p := \{\sum_{j=1}^{|\omega|} S_r^{|\omega|-j}\bar{\eta} \cdot \omega(j) \in A(N \times \{0,1\}, K) :$ $|\omega| \le q \le p$ for $p \in N$, $\bar{\eta} = e_{(0,1)} - e_{(0,0)}\}$. then $A(N \times \{0,1\}, K)^S_p \subset A(N \times \{0,1\}, K)_p$ holds. And i_p is a canonical injection $i_p : A(N \times \{0,1\}, K)^S_p \to A(N \times \{0,1\}, K)_p$.

(8-B.1) Definition

If a free motion with an affine map $((X, F), g^0, g)$ satisfies $X =\ll \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + g \cdot \omega(j)); \omega \in \Omega_p\} \gg$, then $((X, F), g^0, g)$ is said to be p quasi-reachable.
If a free motion with an affine map $((X, F), g^0, g)$ satisfies $X = \{\sum_{j=1}^{|\omega|} F^{|\omega|-j}(g^0 + g \cdot \omega(j)); \omega \in \Omega_p\}$, then $((X, F), g^0, g)$ is said to be p reachable.

Remark 1: Note that $((X, F), g^0, g)$ is p-quasi reachable if and only if $G_p := G \cdot J_p : A(N \times \{0,1\}, K)_p \to X$ is surjective.
Where G is the linear input map : $(A(N \times \{0,1\}, K), S_r) \to (X, F)$ corresponding to $((X, F), g^0, g)$.
Remark 2: Note that $((X, F), g^0, g)$ is p-reachable if and only if $G^R_p := G \cdot J_p \cdot i_p : A(N \times \{0,1\}, K)^S_p \to X$ is surjective.

(8-B.2) Lemma

Let $A^S_{(p,\underline{N}-p)}$ be the partial reachable linear input/output map corresponding to $\underline{a} \in F(\Omega_{\underline{N}}, Y)$. Then the following diagrams commute.

1)

$$\begin{array}{ccc} A(N\times\{0,1\},K)^S_p & \xrightarrow{A^S_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p},Y) \\ \Big\downarrow \underline{i} & & \Big\downarrow \pi \\ A(N\times\{0,1\},K)^S_{p+1} & \xrightarrow{A^S_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1},Y) \end{array}$$

Where $\underline{i}$ is a canonical injection and π is a canonical surjection.

2)

$$\begin{array}{ccc} A(N\times\{0,1\},K)^S_p & \xrightarrow{A^S_{(p,\underline{N}-p)}} & F(N_{\underline{N}-p},Y) \\ S_r \Big\downarrow & & \Big\downarrow \underline{S_l} \\ A(N\times\{0,1\},K)^S_{p+1} & \xrightarrow{A^S_{(p+1,\underline{N}-p-1)}} & F(N_{\underline{N}-p-1},Y) \end{array}$$

[proof] These can be obtained by direct calculation.

(8-B.3) Proposition

Let $A^S_{(p_1,\underline{N}-p_1)}$ be the partial reachable linear input/output map corresponding to $\underline{a} \in F(\underline{N},Y)$ and p_2 be any integers such that $0 \leq p_2 \leq p_1 < \underline{N}$. If im $A^S_{(p_2+1,\underline{N}-p_2)-1} =$ im $A^S_{(p_2,\underline{N}-p_2-1)}$ holds, then im $A^S_{(p_1,\underline{N}-p_1)}$ $=$ im $A^S_{(p_2,\underline{N}-p_1)}$ holds.

[proof] Let $U = \{0,1\}$ in Proposition (7-C.6) of Appendix 7.7. This proposition corresponds to it. Hence, this will be obtained.

(8-B.4) Proposition

Let $A^S_{(\ ,\ )}$ be the partial reachable linear input/output map corresponding to $\underline{a} \in F(\underline{N},Y)$. For p_1 and p_2 be any integers such that $0 \leq p_2 < p_1 < \underline{N}$. If ker $A^S_{(p_1,\underline{N}-p_1)} =$ ker $A^S_{(p_1,\underline{N}-p_1-1)}$ holds, then ker $A^S_{(p_2,\underline{N}-p_1)}$ $=$ ker $A^S_{(p_2,\underline{N}-p_2)}$ holds.

[proof] This is the same as Proposition (7-C.7) in Appendix 7.7.

(8-B.5) Lemma

For a partial linear input/output map $A^S_{(\ ,\ )}$ corresponding to $\underline{a} \in F(\underline{N}, Y)$ and an Almost Linear System $\sigma = ((X, F), g^0, g, h, h^0)$, the next matters hold.

Where $G_p := G \cdot J_p$, $H_q := P_q \cdot H$ for the linear input map G corresponding to g^0, g and the linear output map H corresponding to h. And $A_{(p,q)} := H_q \cdot J_p$.

1) σ is a partial realization of $\underline{a}$ if and only if the following figure commutes for any p such that $0 \le p < \underline{N}$.

2) σ is a natural partial realization of $\underline{a}$ if and only if the following figure commutes, G_p is surjective and $H_{\underline{N}-p-1}$ is injective for some p such that $0 \le p < \underline{N}$.

$$
\begin{array}{ccccc}
A(N \times \{0,1\}, K)^S_p & \xrightarrow{G^S_p} & X & \xrightarrow{H_{\underline{N}-p}} & F(N_{\underline{N}-p}, Y) \\
\downarrow S_r & & \downarrow F & & \downarrow \underline{S_l} \\
A(N \times \{0,1\}, K)^S_{p+1} & \xrightarrow{G^S_{p+1}} & X & \xrightarrow{H_{\underline{N}-p-1}} & F(N_{\underline{N}-p-1}, Y)
\end{array}
$$

[proof] These can be obtained by definition of the partial and natural partial realization.

(8-B.6) Proof of Theorem (8.24)

In Chapter 7, we obtained Theorem (7.24) for Almost Linear Systems as same as this. Based on the theorem, we will prove this theorem. In the same way as the theorem, we will prove this theorem by rewriting the conditions of partial Input/Output Matrix and Hankel matrix in Theorem (8.24) to partial linear input/output map $A_{(\ ,\ )}$ and reachable linear input/output map $A^S_{(\ ,\ )}$ corresponding to $\underline{a} \in F(\underline{N}, Y)$. By using Propositions (7-C.6) and (7-C.7), the conditions of the matrices can be equivalently changed to the following equations (1) to (4):

(1) $\mathrm{im}\ A_{(p,\underline{N}-p-1)} = \mathrm{im}\ A_{(p,\underline{N}-p-1)}$

(2) $\ker\ A_{(p,\underline{N}-p)} = \ker\ A_{(p,\underline{N}-p-1)}$

(3) $\mathrm{im}\ A^S_{(p,\underline{N}-p-1)} = \mathrm{im}\ A^S_{(p+1,\underline{N}-p-1)}$

(4) $\ker\ A^S_{(p,\underline{N}-p)} = \ker\ A^S_{(p,\underline{N}-p-1)}$

Therefore we will prove the theorem by using (1) to (4).
Firstly, we show that the above equations (1) to (4) are necessary.
Let $\sigma = ((X, F), g^0, g, h, h^0)$ be a natural partial realization of $\underline{a} \in F(\underline{N}, Y)$, then σ is p-quasi-reachable and q-observable for some p and q such that $p + q < \underline{N}$. Let G be the linear input map corresponding to g^0, g and H be the linear observation map corresponding to h, and let $p \leq p'$ and $q \leq q'$, then $G_{p'} := G \cdot J_{p'}$ and $G^S_{p'} := G \cdot J_{p'} \cdot i_{p'}$ are onto, $H_{q'} := P_{q'} \cdot H$ is one-to-one. Therefore $A_{(p',q')} := H_{q'} \cdot J_{p'}$ and $A^S_{(p',q')} := H_{q'} \cdot J^S_{p'}$ satisfy conditions (1) to (4).
Next, we show that the equations (1) to (4) are sufficient.
Set $S :=$ ker $A_{(p+1,\underline{N}-p-1)}$ and $Z :=$ im $A_{(p,\underline{N}-p)}$. Then equation (2) implies that a composition map $\pi \cdot j : Z \xrightarrow{j} F(N_{\underline{N}-p}, Y) \xrightarrow{\pi} F(N_{\underline{N}-p-1}, Y)$ is injective. Where π and j are the same as in Proposition (7-C.4) in Chapter 7. Hence Z satisfies the condition 3 in Proposition (7-C.4) of Chapter 7. Equation (1) implies that there exist $e_{(l,u_i)} \in A(N \times \{0,1\}, K)_p$ such that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)}) = A_{(p,\underline{N}-p-1)}(\sum_{l,u_i} \lambda(l, u_i)e_{(l,u_i)})$ for any $u \in \{0,1\}$. By Lemma (7-C.5) of Chapter 7, we obtain that $A_{(p+1,\underline{N}-p-1)}(e_{(p+1,u)} - \sum_{l,u_i} \lambda(l, u_i)e_{(l,u_i)}) = 0$, and $e_{(p+1,u)} - \sum_{l,u_i} \lambda(l, u_i)e_{(l,u_i)} \in S$ holds. This implies that S satisfies the condition 2 in Proposition (7-C.2) of Chapter 7. Let j be the canonical injection : $A_{(p,\underline{N}-p-1)} \to F(N_{\underline{N}-p-1}, Y)$ and π is the same as in Proposition (7-C.4) of Chapter 7, $B := \overline{(j)}^{-1} \cdot \pi \cdot j : Z \to$ im $A_{(p,\underline{N}-p-1)}$ is a bijective linear map by (2) in Proposition (7-C.2) of Chapter 7. When we consider the bijective linear map $A^b := A^b_{(p+1,\underline{N}-p-1)} : A(N \times \{0,1\}, K)_{p+1}/S \to$ im $A_{(p+1,\underline{N}-p-1)}$ associated with $A_{(p+1,\underline{N}-p-1)} : A(N \times \{0,1\}, K)_{p+1} \to F(N_{\underline{N}-p-1}, Y)$, equation (2) implies that a linear map $B^{-1} \cdot A^b$ is a bijective linear map : $A(N \times \{0,1\}, K)_{p+1}/S \to Z$. For any $\lambda \in A(N \times \{0,1\}, K)_p \to S$, $A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by injection of $B^{-1}A^b$. Hence $A_{(p+1,\underline{N}-p-1)}(S_r\lambda) = \underline{S_l}A_{(p,\underline{N}-p)}(\lambda) = 0$ holds by using 2) in Lemma (7-C.5) of Chapter 7. This implies that $S_r\lambda \in S$. Therefore S satisfies condition 1 in Proposition (7-C.2) of Chapter 7. Then Proposition (7-C.2) implies that a free motion with an affine map $(A(N \times \{0,1\}, K)_{p+1}/S, \underline{S_r}), [e(0,0)], [\eta])$ is p-quasi-reachable. Where $[\eta] = e_{(0,1)} - e_{(0,0)} + S$.
Here equation (1) implies that there exists $x \in$ im $A_{(p,\underline{N}-p-1)}$ such that $\underline{j}(x) = \underline{S_l} \cdot j(z)$ for any $z \in Z$. Moreover, by surjection of B, there exists $z' \in Z$ such that $B(z') = x$. Hence, $\underline{S_l} \cdot j(z) = \underline{j}(x) = \underline{j} \cdot B(z') = \pi \cdot j(z')$, which implies that im $(\underline{S_l} \cdot j) \subseteq$ im $\overline{(\pi \cdot j)}$. It follows that Z satisfies condition 4 in Proposition (7-C.4) of Chapter 7 and $((Z, F), 0)$ is $(\underline{N} - p - 1)$-

observable. Where F is given by the equation $Fz = (\pi \cdot j)^{-1} \cdot \underline{S_l} \cdot j(z)$ for any $z \in Z$. We can also show that $B^{-1} \cdot A^b$ is a free motion morphism $: (A(N \times \{0,1\}, K)_{p+1}/S, \underline{S_r}) \to (Z, F)$, and that an Almost Linear System $\sigma_1 = (A(N \times \{0,1\}, K)_{p+1}/S, \underline{S_r}), [e(0,0)], [\eta], 0 \cdot B\ -1 \cdot A^b, a(1))$ is isomorphic to an Almost Linear System $\sigma_2 = ((Z, F), B^{-1} \cdot A^b \cdot [e(0,0)], B^{-1} \cdot A^b \cdot [\eta], 0, a(1))$. It follows that σ_1 and σ_2 are the natural partial realizations of $\underline{a} \in F(\underline{N}, Y)$. Hence, the natural partial realizations of $\underline{a} \in F(\underline{N}, Y)$ exist. By the equation (3), σ_1 is clearly p-reachable.

(8-B.7) Proof of Theorem (8.25)
Let $\sigma_2 = ((Z, F), B^{-1} \cdot A^b \cdot [e(0,0)], B^{-1} \cdot A^b \cdot [\eta], 0, a(1))$ be an intrinsically Almost Linear System introduced in (8-B.6) which is a proof of Theorem (8.24). And let $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, h^0)$ be the reachable standard system which is isormorphic to σ_2. Moreover, let T be an Almost Linear System morphism $T : \sigma_s \to \sigma_2$. Then $Te_i = \underline{S_l}^{i-1}\underline{\eta}$ for $i(1 \le i \le n)$ and $Tg_s^0 = \eta^0$ hold. Then the condition which implies that there exists $x^0 \in K^n$ such that $[F_s - I]x^0 = g_s^0$ is equivalent to there exists $x = [x_1, x_2, \cdots, x_n]^T$ such that $[\underline{a}(0) - \underline{a}(1), \underline{a}(0^2) - \underline{a}(0), , \underline{a}(0^p) - \underline{a}(0^{p-1})]^T = (\sum_{i=1}^n x_i(\underline{S_l}^i\underline{\eta} - \underline{S_l}^{i-1}\underline{\eta})$ in Z. Adding to the result of Theorem (8.24), we obtain this theorem.

(8-B.8) Proof of Theorem (8.26)
In Theorem (7.25) of Chapter 7, we proved the same for Almost Linear Systems. We have already shown that So-called Linear Systems are in a subclass of Almost Linear Systems. We have obtained cleared relations between So-called Linear Systems and Almost Linear Systems in Proposition (8.9). Therefore, this theorem is obtained from the these facts.

(8-B.9) Proof of Theorem (8.27)
Let's consider the natural partial realization $\sigma_2 = ((Z, F), B^{-1} \cdot A^b \cdot [e(0,0)], B^{-1} \cdot A^b \cdot [\eta], 0, a(1))$ of $\underline{a} \in F(\underline{N}, Y)$ given in (8-B.6). Then we can obtain the reachable standard system $\sigma_s = ((K^n, F_s), g_s^0, g_s, h_s, a(1))$ from σ_2 in the same manner as Theorem for a Realization Procedure (7.21) (also see (7-B.19)).

8.6.C Real-Time Partial Realization of So-Called Linear Systems

(8-C.1) Proof of Lemma (8.30)

Note that jth component of column vectors $\underline{S_l}^i(\underline{\chi}^0) \in K^L$ satisfies $(\underline{S_l}^i(\underline{\chi}^0)(j-1) = \underline{a}(0^{i+j}) - \underline{a}(0^{i+j-1})$ for any $j \in 1, 2, \cdots, L-i$. Then $\{\underline{a}(0^k|\omega_1)$; any $k \in N\}$ can be uniquely determined by linearly dependence. Since $\underline{a}$ is time-invariant, $\underline{a}(0^{k-1}|1\omega_1) - \underline{a}(0^k|\omega_1) + \underline{a}(0^k) - \underline{a}(0^{k-1}) = \underline{a}(0^{k-1}|1) - \underline{a}(0^{k-1}) = (\underline{S_l}^i(\underline{\chi}^0 + \underline{\chi}))(j-1)$ can be obtained for $i + j = k$. Therefore, if we add a further input $\omega_2 = 0^{L_2+L-1}|1$, step 2) can be inspected. Since jth component of column vectors $\underline{S_l}^i(\underline{\chi}^0 + \underline{\chi}) \in K^{L-1}$ is $(\underline{S_l}^i(\underline{\chi}^0 + \underline{\chi})(j-1) = \underline{a}(0^{i+j}|1|\omega_1) - \underline{a}(0^{i+j}|\omega_1) + \underline{a}(0^{i+j}) - \underline{a}(0^{i+j-1})$, $\{\underline{a}(0^k|\omega_2|\omega_1)$; any $k \in N\}$ can be uniquely determined by linear dependence.

Then a partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ is obtained. Since the physical object is less than L dimensional, the partial Input/Output Matrix $(I/O)_{\underline{a}\ (L-1,p)}$ contains the whole input/output data as in (8.24).

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